SOME NONSTATIONARY SUBDIVISION SCHEMES FOR DESIGNING CURVES AND SURFACES

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Dedicated to my parents

CERTIFICATE

It is certified that the work contained in the thesis entitled *Some Nonstationary Subdivision Schemes for Designing Curves and Surfaces*, by *Mahendra Kumar Jena*, has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

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Synopsis

This thesis introduces new nonstationary subdivision schemes for designing curves and surfaces. Any subdivision scheme for generation of curves consists of the following. Initially a set of points called control points such as

$$P^0 := \{p_\alpha \in \mathbb{R}^s \ : \ \alpha \in \mathbb{Z}\}, \ s = 1, 2 \text{ or } 3$$

is known. Any subdivision scheme $\{S_{a_k}\}$ iteratively generates a new set of control points

$$P^k := \{p^k_\alpha \in \mathbb{R}^s \ : \alpha \in \mathbb{Z}\}$$

at the kth level by a subdivision rule:

$$p^k_\beta = (S_{a_k} P^{k-1})_\beta = (S_{a_k} S_{a_{k-1}} \cdots S_1 P^0)_\beta = \sum_{\alpha \in \mathbb{Z}} a^{(k)}_{\beta-2\alpha} \, p^k_\alpha, \quad \beta \in \mathbb{Z}.$$

The set $\mathbf{a}^{(k)} = \{a^{(k)}_i \ : \ i \in \mathbb{Z}\}$ of coefficients is called the mask at kth level of the subdivision scheme. If the set $\mathbf{a}^{(k)}$ is such that its elements depend upon k, then the associated subdivision scheme is called a nonstationary subdivision scheme.

Since the scheme is applied componentwise, one restricts the study to control points on $\mathbb{R}$. Therefore, the initial data $\{p^0_\alpha \in \mathbb{R}, \ \alpha \in \mathbb{Z}\}$ as well as the scheme $\{S_{a_k}\}$ are considered on $\mathbb{R}$ only.

The thesis consists of five chapters. The first chapter presents a brief review of subdivision schemes and some basic mathematical preliminaries. In Chapters 2 to 4 three different schemes for designing curves are presented. For each scheme a subdivision algorithm is constructed, its convergence analysis is studied and some applications and advantages are outlined. Convergence of all the schemes is shown using a theorem due to Dyn and Levin in [35]. In Chapter 5, a nonstationary subdivision scheme is presented for generating smooth surfaces where the data points are not necessarily given on a rectangular grid.

In **Chapter 2**, we present a subdivision scheme for the generalized Bernstein Bezier curves [38]. The scheme is a generalization of Lane and Riesenfeld algorithm for classical Bezier curves [49]. As particular cases one can obtain subdivision schemes for classical as well as trigonometric Bezier curves [48]. We also point out the advantages of using the subdivision scheme for designing curves as well as surfaces. The algorithm is given as follows.

Algorithm 0.1. *Starting from q_i^0, $i = 0, 1, \ldots, n$ the set of control points q_i^1, $i = 0, 1, \ldots, 2n$ are obtained as follows:*

Let $p_i^0 = q_i^0$, $i = 0, 1, \ldots, n$.

For $k = 1, \ldots, n$, set

$$p_i^k = b_0(l)\, p_{i-1}^{k-1} + b_1(l)\, p_i^{k-1}, \quad i = k, k+1, \ldots n.$$

Define

$$q_i^1 = \begin{cases} p_i^i, & i = 0, 1, \ldots, n, \\ p_n^{2n-i}, & i = n, n+1, \ldots, 2n. \end{cases}$$

In the above algorithm $b_0(l) = \frac{d(-\frac{l}{2})}{d(-l)}$ and $b_1(l) = \frac{d(\frac{l}{2})}{d(l)}$ where $d(x)$ is the solution of an initial value problem [38]. Besides the algorithm we also give some useful properties of the function $d(x)$ which generates the space of generalized Bernstein polynomials and some results concerning generalized convex hull, called $\mathcal{D}$-convex hull associated with $d(x)$.

There is extensive work involving trigonometric spline as a tool for designing curves. Successful efforts have been made in [48] to establish algorithms for trigonometric splines analogous to algorithms for polynomial splines of one variable. There is an algorithm in literature for trigonometric splines [48] analogous to Oslo algorithm. However, there is no algorithm for trigonometric splines corresponding to Lane-Riesenfeld algorithm of polynomial spline. In **Chapter 3** we introduce an analog of Lane-Riesenfeld algorithm for the trigonometric spline with uniform knots. It turns out to be a non-stationary binary subdivision scheme.

Algorithm 0.2. *Let $n \geq 1$ and $n = 2k - 1$ or $n = 2k$ for some $k \geq 1$. Starting from q_i^0, $i = 0, 1, \ldots, n$ the first level control points q_i^1, $i = n - 1, \ldots, 2m + 1$ are obtained as*

follows: Set

$$p_{2i}^1 = \begin{cases} q_i^0, & i = 0,1,\ldots,m \quad \textit{if } n = 2k-1 \\ \frac{s(\frac{l}{2})}{s(l)}\,(q_{i-1}^0 + q_i^0), & i = 1,2,\ldots,m \quad \textit{if } n = 2k \end{cases}$$

$$p_{2i+1}^1 = q_i^0, \qquad i = 0,1,\ldots,m \quad (n = 2k \textit{ or } n = 2k-1).$$

For $j = 2, 3, \ldots k$, *define*

$$p_i^j = \begin{cases} w(2j-3,l)\,\{w(2j-2,l)\,p_{i-2}^{j-1} + p_{i-1}^{j-1} + w(2j-2,l)\,p_i^{j-1}\}, \\ \qquad \textit{for } n = 2k-1, \textit{ and } i = 2j-2,\ 2j-1,\ldots,2m+1. \\ w(2j-2;l)\,\{w(2j-1;l)\,p_{i-2}^{j-1} + p_{i-1}^{j-1} + w(2j-1;l)\,p_i^{j-1}\}, \\ \qquad \textit{for } n = 2k, \textit{ and } i = 2j-1,\ 2j,\ldots,2m+1. \end{cases}$$

Set $q_i^1 = p_i^k$, $i = n-1, \ldots, 2m+1$.

In the above algorithm $w(j;l) = \frac{\sin(jl/2)}{\sin(jl)}$. The basic mathematical contribution consists of an introduction of a convolution formula for trigonometric B-splines. The above scheme helps in the exact reconstruction of cosine, sine functions and circles.

In **Chapter 4**, we introduce a nonstationary subdivision scheme whose limit curve interpolates a given set of data points.

Algorithm 0.3. *Given the control points* $\{p_i^0 \in \mathbb{R},\ i = -2, -1, \ldots, n+2\}$ *the control points* $\{p_i^{k+1},\ i = -2, -1, 0, \ldots, 2^{k+1}n + 1\}$ *at level* $k+1$ *are obtained by the following recursive relation:*

$$p_{2i}^{k+1} = p_i^k, \qquad -1 \le i \le 2^k n + 1$$

$$p_{2i+1}^{k+1} = -w_k p_{i-1}^k + (\frac{1}{2} + w_k)p_i^k + (\frac{1}{2} + w_k)p_{i+1}^k - w_k p_{i+2}^k, \qquad -1 \le i \le 2^k n$$

where $w_k = \dfrac{\sin^2(\frac{l}{2^{k+2}})}{2\sin(\frac{l}{2^k})\sin(\frac{l}{2^{k+1}})}$.

This scheme is a generalization of a well known four point subdivision scheme of Dyn and Levin [32]. We establish that our scheme is asymptotically equivalent to the four point scheme of Dyn and Levin [35]. One of the important features of this scheme is that it reproduces functions 1, $\cos(\alpha x)$, $\sin(\alpha x)$ and their linear combinations. In particular if we choose vertices of a regular n-gon as initial control points, then the limit curve turns out to be the original circle passing through the vertices of the regular

n-gon. We also show that this scheme can be used to approximate smooth functions quadratically.

The above schemes for curves are generalized in a natural way to nonstationary subdivision schemes for tensor product surfaces. These schemes require that the data set should lie in a rectangular grid. However, quite often the data set does not lie on a rectangular grid. There are several stationary subdivision schemes like ones given by Doo-Sabin [26] and Catmull-Clark [16] which generate surfaces from such data. However, there are only few results available dealing with nonstationary subdivision schemes generating surfaces for such data except the work of Dyn and Levin [35] which deals with tensor product exponential splines and exponential box splines. Moreover, formulating nonstationary schemes for arbitrary data and their convergence analysis is a difficult task since the masks keep changing in each iteration. In **Chapter 5**, we present a nonstationary subdivision scheme generating surfaces for data prescribed on an arbitrary topology. The scheme can be viewed as a generalization of the subdivision scheme for the trigonometric splines introduced in Chapter 3 to cases with arbitrary surface data. The scheme is formulated along the line of the Doo-Sabin and Catmull-Clark schemes and convergence analysis follows the mechanism of U. Reif [63] and Jorg Peters [59].

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Chapter 1

Introduction

1.1 A Brief History

Computer Aided Geometric Design or CAGD in short evolved during 1960s as a result of development of digital computers and simultaneous development of software for designing curves and surfaces in designing industry. These new tools to a large extent replaced expensive methods of building models and templates. Therefore the focus shifted to find computer compatible forms of curves and surfaces by replacing the master model by a mathematical model, a computerized form of 3D object. This resulted in defining a surface as a set of triplets in $\mathbb{R}^3$ and a smooth injective map from $\mathbb{R}^2$ to $\mathbb{R}^3$.

The parametric curves and surfaces which were till then exclusively studied by differential geometry [13] were subjected to numerical studies on the basis of their numerical construction ushering a new area of study called Computer Aided Geometric Design formally so designated in 1974 [5].

In surface reconstruction, one is provided with a large number of point coordinates. This information is obtained from the drawing board or from the physical model. There exist several methods to generate surfaces from this set of discrete points. Some of the important methods like the Bezier methods, B-spline methods, and subdivision methods are briefly outlined below.

Bezier methods:
The Bezier methods of generating curves and surfaces were developed independently by P. de Casteljau about 1959 at Citroen [14, 15] and by P. Bezier about 1967 at Renault

[6, 7, 8, 9]. The work of Bezier which was titled UNISURF was published earlier than the Citroen technical report and hence the theory now bears the name of Bezier.

Bezier curves and surfaces provide a powerful tool in curves and surfaces design: They are affinely invariant, stable and have linear precision property. Moreover the Bezier curve has variation diminishing property.

In theory of Bezier curves barycentric coordinates and Bernstein polynomials play a key role. This encouraged others to define barycentric like coordinates, and define Bernstein polynomials and Bezier curves on these newly defined objects. Recently, Alfeld, Neamtu and Schumaker [1, 2] introduced barycentric like coordinates for circular and spherical coordinates. This helped to introduce circular Bernstein-Bezier or CBB curves and spherical Bernstein-Bezier or SBB patches. The CBB curve is also called as trigonometric Bernstein-Bezier or TBB curve by Koch et.al.[48]. Before them Brown and Worsey [12] had tried to define barycentric coordinates on sphere. Recently Gonsor and Neamtu [38] presented a unified Bernstein-Bezier theory via the concept of a polar form. They consider a class of functions (here we call such functions generalized Bernstein-Bezier or GBB polynomials) which include the classical Bernstein Bezier polynomials, trigonometric Bernstein Bezier polynomials as special cases. The associated barycentric coordinates are defined as solution of an initial value problem of a second order differential equation. The concept of Bernstein-Bezier polynomial is now very broad which contains polynomials as well as non polynomials.

B-spline methods:

Study of splines started with I.J. Schoenberg in 1946 as univariate piecewise polynomial functions [67, 68] and has been extensively studied in the theory of approximations since then.

Apparently the earliest use of spline in CAGD was about 1963 by J. Ferguson [37] in Boeing Company. He used cubic splines to interpolate a given data. At about same time C. de Boor and W. Gordon studied these curves at General Motors. Use of splines became popular in CAGD after the discovery of compactly supported basis function called B-splines. The name B-spline was first used by Schoenberg in 1967 [70]. Schoenberg first dealt with B-spline on equispaced knots and defined them via a convolution formula [67]. This convolution formula helped to find an efficient algorithm, known as Lane and Riesenfeld algorithm for univarite spline curve. Gordon [39] and Riesenfeld [65] first used the B-splines in the context of parametric curve design and

showed that they are proper generalization of Bezier curves. A B-spline can also be expressed as a piecewise Bezier form [42].

The spline curves have most of the properties of Bezier curves: They are affinely invariant, variation diminishing and possess linear precision property. In addition to that B-spline curves have local control property.

Since the discovery of splines many successful attempts have been made to generalize B-splines, involving class of functions which are not polynomials such as rational splines, trigonometric splines, Chebychev splines and L-splines etc. A comprehensive theory on these non-polynomial splines can be found in the book [71] by Schumaker. The contemporary meaning of spline is very broad and it includes general piecewise functions rather than piecewise polynomials only.

Recently trigonometric splines were given special attention for designing curves. Trigonometric splines were introduced by I.J.Schoenberg in [69], and have been extensively studied in the literature [47, 48, 54, 55, 71, 72, 74]. It has been observed that most features of classical polynomial splines carry over to trigonometric splines, in particular, for example, it is shown that any trigonometric spline is a linear combination of trigonometric B-splines [55, 71]. Besides, a recurrence relation for trigonometric B-spline has been established in [55]. In [48], the notion of control curve for a trigonometric spline has been introduced and studied, besides the convex hull property and variation diminishing property of trigonometric splines are also established in the spirit of the results available for polynomial splines.

The simplest procedure to obtain two dimensional splines is to use tensor product for rectangular regions. The tensor product splines inherits all the properties of univariate splines except variation diminishing property. A more general approach to the construction of multivariate splines consists of simplex splines [10], polyhedral splines [22] and box splines [11]. Multivariate splines are extensively studied in [20, 57].

Subdivision methods:

Subdivision methods consist of classes of numerically stable easily implementable algorithms for the generation of parametrized curves and surfaces. These algorithms are sometimes referred to as corner cutting algorithms. Essentially, a subdivision method describes a continous shape as a limit of a sequence of discrete shapes described by so called control points. These discrete shapes are obtained by linear transformations based on splitting and averaging.

Subdivision methods are also faster and easier methods to construct Bezier curves and surfaces as well as spline curves and surfaces. Moreover subdivision schemes are also used to generate surfaces from a data set associated with an arbitrary topology in a very efficient way unlike other available methods. They are also intimately linked to multiresolution structures and are used widely in other areas of computer graphics such as constructing wavelets and fractals, animation and rendering etc. Recently subdivision schemes are applied to solve problems in variational calculus [45, 75], ordinary differential equations [76, 78] and to approximate functions and convex compact sets [30].

The general set up for design of surfaces via the subdivision method is the following: The designer starts from a finite set of points, called **control points** and a **connectivity rule** between them. The points are arranged according to the univariate index set $P_1, P_2, \ldots, P_n \in \mathbb{R}^2$ (or $\mathbb{R}^3$). Then more and more control points are generated by a **subdivision rule**. At each iteration level a **control polygon** is identified with the control points. If the subdivision rule is suitably chosen, the sequence of control polygons converge to a smooth limit function in the sense that the limit function has the similar shape as the control polygon and is very close to them after certain iterations. The curve can also be made to pass through the original control points. In this case, the schemes are called interpolatory.

The general set up for surface designing in the subdivision method is the following: We start with a finite set of **control points** in $\mathbb{R}^3$ and a **connectivity rule** among them. The connectivity rule associates the control points to vertices of edges, faces and hence associates them to a **control polyhedron**. The **oder of a face** is the number of vertices constituting the face. The **order of a vertex** is the number of faces associated with the vertex. The control polyhedron is said to have a **regular topology** if the control points have an indexing

$$p_\alpha \in \mathbb{R}^3,\ \alpha \in \mathbb{Z}^2$$

and order of all the faces, order of all the interior vertices and order of all the boundary vertices are same respectively. The regular topology contains a tensor product topology as well as a triangular topology. Then a **subdivision rule** is used iteratively for generating new sets of control points and corresponding control polyhedrons (see Fig 1.1). The rule is such that the sequence of control polyhedrons converges to a smooth limit surface.

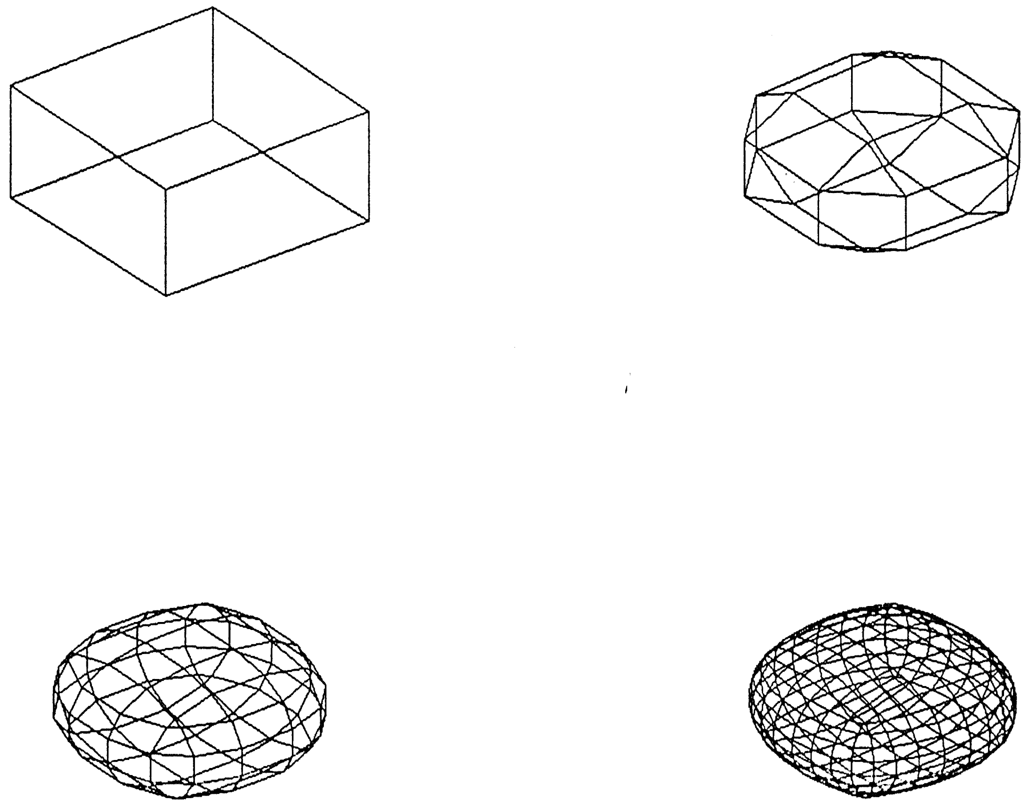


Figure 1.1: Catmull and Clark subdivision scheme

In a subdivision scheme, if the subdivision rule used is same at all levels of iterations hen the subdivision scheme is called a **stationary** subdivision scheme otherwise it is called a **nonstationary** subdivision scheme.

There are various stationary subdivision schemes in the literature. One of the earliest stationary subdivision schemes for curves was studied by G. de Rham [64]. In 1974 G. Chaikin presented a subdivision scheme for quadratic spline curve [19]. Lane and Riesenfeld generalized the Chaikin algorithm and presented a subdivision scheme for polynomial spline curve of arbitrary degree [49]. The latter subdivision scheme is considered as one of the important stationary subdivision schemes in CAGD. Another important stationary subdivision scheme for design of curves is the 4-point interpolatory scheme by N.Dyn [32](see also [27]).

Stationary subdivision schemes for curves are extended to generate tensor product surfaces in a natural way if the control points are arranged in a rectangular topology.

For triangular topology there are also subdivision schemes such as line averaging algorithms [24], subdivision schemes for box splines [23, 43] and butterfly scheme [33]. All the above subdivision schemes are studied by a unified theory as given in the work of Cavaretta and Micchelli [17] and Dyn [29](see also [18, 31, 56]).

Stationary subdivision schemes for arbitrary topologies were first studied in 1978 by Catmull and Clark [16], Doo and Sabin [26], and later by Loop [52]. Ball and Storry [3, 4] analyzed these schemes using a matrix approach. In 1995, U. Reif presented a theory for these schemes [62, 63]. He provided a necessary and sufficient condition for the convergence of such schemes ensuring tangent plane continuity of the limit surface. His study was based on the nature of eigenvalues of the associated local subdivision matrix. After this, a stream of new schemes like mid-edge subdivision scheme [58], Kobbelt subdivision scheme [44, 46, 73] , tagged subdivision scheme [41] and combined subdivision scheme [50, 51] were introduced. At the same time new approaches to analyze these schemes were also introduced [59, 60, 61].

Recently several nonstationary subdivision schemes for curves have been introduced. The first nonstationary subdivision scheme for curves was proposed by Rvachev [66]. In [35], Dyn and Levin presented a general theory for a special class of nonstationary subdivision schemes, called asymptotically equivalent subdivision schemes, where, as an example they presented nonstationary subdivision schemes for exponential splines and exponential box splines. A nonstationary scheme generating circles is also presented by Dyn in [34]. Nonstationary subdivision schemes are also recently used by J. Warren to solve problems in variational calculus [75] and by H. Weimer to solve ordinary differential equations [77, 78] and physical problems in fluid dynamics [76].

1.2 Contribution of the Thesis

In **Chapter 2** of the thesis, we present a subdivision scheme for the generalized Bernstein-Bezier curves. This scheme is a generalization of Lane and Riesenfeld algorithm for classical Bezier curves [49]. As particular cases of the above scheme, one gets the subdivision schemes for classical as well as trigonometric Bezier curves. We also highlight certain advantages of using this subdivision scheme for designing curves as well as surfaces. Besides the algorithm we also describe some useful properties of the function $d(x)$ which generates the space of generalized Bernstein-Bezier polynomials. Besides, some results dealing with generalized convex hull, called, $\mathcal{D}$-convex hull

associated with $d(x)$ is also discussed.

Recently researchers working in the area of designing of curves and surfaces have started looking trigonometric spline as a tool for designing purposes. Successful efforts have been made [48] to establish algorithms for trigonometric splines analogous to algorithms available for polynomial spline functions of a single variable. There is an algorithm in literature for trigonometric splines [48] generalizing Oslo algorithm. However, there is no subdivision algorithm for trigonometric splines corresponding to Lane-Riesenfeld algorithm of polynomial splines. In **Chapter 3** we introduce an analog of Lane-Riesenfeld algorithm for the trigonometric splines with uniform knots. A nonstationary binary subdivision scheme is introduced to generalize this algorithm. The basic methodological contribution in this Chapter is the introduction of a convolution formula for trigonometric splines and showing the probability of exact reconstruction of parametric curves $(\cos(x), \sin(x))$ or circles.

In **Chapter 4**, we introduce a nonstationary subdivision scheme whose limit curve interpolates a given set of data points. This scheme is a generalization of a well known four point subdivision scheme of Dyn and Levin [32]. We establish that our scheme is asymptotically equivalent to the four point scheme of Dyn and Levin [35]. One of the important features of this scheme is that it reproduces functions 1, $\cos(x)$, $\sin(x)$ and their linear combinations. In particular if we choose vertices of a regular n-gon as initial control points, then the limit curve is the original circle passing through the vertices of the regular n-gon. We also show that this scheme can be used to approximate smooth functions effectively.

There is not much literature available dealing with nonstationary subdivision schemes for the generation of surfaces except the work of Dyn and Levin [35] which presents nonstationary schemes concerning tensor product exponential splines and exponential box splines. In **Chapter 5** of the thesis, scheme for trigonometric splines introduced in Chapter 3 is generalized into a nonstationary subdivision scheme generating surfaces from tensor product topologies as well as arbitrary topologies. The subdivision scheme is in some sense a generalization of the Doo-Sabin scheme and the Catmull-Clark scheme. The limit surface retains same shape as the limit surfaces of these stationary schemes, but the sizes of the limit surfaces can be varied by changing certain design parameter or functions.

1.3 Mathematical Preliminaries

In this section we briefly outline some relevant definitions and results on subdivision schemes. First we consider subdivision schemes concerning a regular topology. We mainly concentrate our discussion on nonstationary subdivision schemes.

Given a set of control points

$$P^0 := \{p_\alpha \in \mathbb{R}^s : \alpha \in \mathbb{Z}^d\},\ d = 1 \text{ or } 2, s = 1, 2 \text{ or } 3$$

at level 0, a subdivision scheme $\{S_{a_k}\}$ generates a new set of control points

$$P^k := \{p_\alpha^k \in \mathbb{R}^s : \quad \alpha \in \mathbb{Z}^d\}$$

at the kth level by a subdivision rule:

$$p_\beta^k = (S_{a_k} P^{k-1})_\beta = (S_{a_k} S_{a_{k-1}} \cdots S_1 P^0)_\beta = \sum_{\alpha \in \mathbb{Z}^d} a_{\beta-2\alpha}^{(k)} p_\alpha^{k-1}, \quad \beta \in \mathbb{Z}^d. \tag{1.1}$$

The set $\mathbf{a}^{(\mathbf{k})} := \{a_i^{(k)} : i \in \mathbb{Z}^d\}$ of coefficients is called the mask at kth level of the subdivision scheme. If the mask is independent of k then the scheme is called stationary otherwise it is called nonstationary. To each subdivision scheme $\{S_{a_k}\}$ defined by the masks $\{\mathbf{a}^{(\mathbf{k})}\}_{k \geq 1}$, we assign kth level characteristic multinomial (polynomial)

$$\mathrm{a}^{(\mathrm{k})}(z) = \sum_{i \in \mathbb{Z}^d} a_i^{(k)} z^i, \quad k \geq 1$$

called the kth level characteristic polynomial of the algorithm.

As the schemes of the type $\{S_{a_k}\}$ are applied componentwise, it is enough to study subdivision schemes for the initial data $P^0 = \{p_\alpha^0 : \alpha \in \mathbb{Z}^d\}$. In these subdivision schemes we associate the points p_α^k, $k \in \mathbb{Z}^d$ to the location values $\frac{\alpha}{2^k}$, $\alpha \in \mathbb{Z}^d$.

Definition 1.1. [35] A subdivision scheme $\{S_{a_k}\}$ is said to be C^m if for every initial data $P^0 \in l^\infty$ there exists a limit function $f \in C^m(\mathbb{R}^d)$ such that

$$\lim_{k \to \infty} \sup_{\alpha \in \mathbb{Z}^d} |p_\alpha^k - f(2^{-k}\alpha)| = 0 \tag{1.2}$$

and $f \not\equiv 0$ for some initial data P^0.

Definition 1.2. [35] A subdivision scheme $\{S_{a_k}\}$ is called stable if there exists a constant M such that

$$\left\|S_{a_{k+m}} \dots S_{a_k}\right\|_\infty \leq M \quad \forall m, k \in \mathbb{N}. \tag{1.3}$$

The limit function F obtained by the application of the scheme $\{S_{a_k}\}$ on the initial data

$$p_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0 \end{cases}$$

is called the *basic limit function* of $\{S_{a_k}\}$. Here 0 refers to the zero element of $\mathbb{Z}^d$.

The following Theorem, which is immediate from Theorem 13 and Lemma 14 of [35] is one of the basic convergence characterizations for general nonstationary subdivision schemes.

Theorem 1.1. *Let a sequence of continuous functions $\{\psi_m : \mathbb{R}^d \to \mathbb{R}\}$, $m \geq 0$ satisfy*

$$\begin{aligned} &(a)\ \lim_{m\to\infty} \sum_{j\in\mathbb{Z}^d} \psi_m(\cdot - j) = 1 \quad \textit{uniformly in } \mathbb{R}^d \\ &(b)\ \|\psi_m\|_\infty \leq B \quad \textit{for some constant } B \\ &(c)\ \psi_m,\ m \geq 0 \quad \textit{are compactly supported.} \end{aligned} \tag{1.4}$$

If the non-stationary scheme $\{S_{a_k}\}$ is C^0 and L is its limit function for the initial data P^0 then the sequence of functions $\{f^k : \mathbb{R}^d \to \mathbb{R}\}$ given by

$$f^k(x) = \sum_{j\in\mathbb{Z}^d} p_i^k \psi_k(2^k x - i), \tag{1.5}$$

converges uniformly to L on the support of L.

Conversely, if a sequence of functions $\{\psi_m \in C(\mathbb{R}^d)\}$ satisfies the stability condition:

$$c\|g\|_\infty \leq \|\sum_{i\in\mathbb{Z}^d} g_i \psi_k(\cdot - i)\|_\infty, \quad g \in l^\infty(\mathbb{Z}^d),\ k \geq K \tag{1.6}$$

for some $K \in \mathbb{Z}_+$ and the sequence of functions $\{f^k\}$ converges uniformly to a continuous function f^∞ on compact subsets of R^d, then the BSS $\{S_{a_k}\}$ is a C^0 convergent subdivision scheme. Moreover $\{S_{a_k}\}$ is C^m if $f^\infty \in C^m(\mathbb{R})$.

In particular if $d = 1$ and $\psi_m : \mathbb{R} \to \mathbb{R}$, $m \geq 0$ are given by

$$\psi_m(t) = (1 - |t|)_+, \tag{1.7}$$

where $(\ )_+$ denotes the plus function or the truncated error function, then ψ_m is compactly supported and clearly satisfies $(a), (b), (c)$ of (1.4). The functions $\{f^k\}$ represents the sequence of control polygons joining the points of P^k. An obvious consequence of

the above Theorem is that if $\{S_{a_k}\}$ is C^0 then the sequence of control polygons $\{f^k\}$ converges uniformly to L. In practice, taking infinite number of iterations is not possible, therefore, the import of the Theorem is that the polygon f^k after certain number (say k) of iterations can be taken as an approximate representation of L. In case $d = 2$, analogous polygons are used for representing the limit surface [40].

Moreover, ψ_m, $m \geq 0$ satisfy (1.6). Therefore if the sequence $\{f^k\}$ associated with this ψ_m converges uniformly then the subdivision scheme $\{S_{a_k}\}$ converges. This is one way of showing convergence of a subdivision scheme.

Let E_d denote the set of extreme points of a unit d-cube. In particular $E_1 = \{0, 1\}$ and $E_2 = \{(0,0), (0,1), (1,0), (1,1)\}$. Let us define the matrix norm corresponding to the scheme $\{S_{a_k}\}$ by

$$\|S_{a_k}\|_\infty := \max\{\sum_{\beta\in\mathbb{Z}^d} |a^{(k)}_{\alpha-2\beta}| : \alpha \in E_d\} \tag{1.8}$$

and define the concept of asymptotic equivalence as follows:

Definition 1.3. [35] Two subdivision schemes $\{S_{a_k}\}$ and $\{S_{b_k}\}$ are asymptotically equivalent if

$$\sum_{k=1}^{\infty} \|S_{a_k} - S_{b_k}\|_\infty < \infty. \tag{1.9}$$

Moreover if $\{S_{b_k}\}$ is a stationary scheme then $\{S_{a_k}\}$ is called an asymptotically stationary scheme.

Let us define the support of a mask $\mathbf{a}^{(k)}$ as the set $\{\alpha \in \mathbb{Z}^d,\ a^k_\alpha \neq 0\}$. As the following Theorems show the convergence of two asymptotically equivalent schemes are closely related.

Theorem 1.2. [35]

(a) Let $\{S_{a_k}\}$ and $\{S_{b_k}\}$ be two asymptotically equivalent schemes. Then $\{S_{a_k}\}$ is C^0 and stable if and only if $\{S_{b_k}\}$ is C^0 and stable.

(b) If $\{S_{a_k}\}$ is asymptotically equivalent to a C^0 stationary scheme $\{S_a\}$ with a finite mask then $\{S_{a_k}\}$ is C^0 and stable.

Theorem 1.3. [35] *Let $\{S_{a_k}\}$ and $\{S_a\}$ be two asymptotically equivalent subdivision schemes having finite masks of same support. If $\{S_a\}$ is C^m and*

$$\sum_{k=0}^{\infty} 2^{mk} \|S_{a_k} - S_a\|_\infty < \infty \tag{1.10}$$

then the non-stationary BSS $\{S_{a_k}\}$ is C^m.

The following examples provide some nonstationary subdivision schemes for curves.

Examples 1.1. (Exponential B-spline) [35] The non-stationary subdivision scheme $\{S_{a_k}\}$ having kth level characteristic polynomial

$$\mathrm{a}^{(k)}(z) = 2\prod_{j=1}^{m}\frac{1}{2}(1+e^{c_j 2^{-k-1}}z), \quad c_j \in \mathbb{C} \tag{1.11}$$

is a C^{m-2} convergent scheme. Moreover the basic limit function F is the exponential B-spline with integer knots, spanned by $\{e^{c_j x};\ j = 1,\ldots,m\}$ and supported on $(0, m)$.

Examples 1.2. (Circle generating scheme) [34] The scheme is represented by the characteristic polynomial

$$\mathrm{a}^{(k)}(z) = \frac{1}{2(1+\cos(\alpha_k))}(1+z)(1+e^{i\alpha_k}z)(1+e^{-i\alpha_k}z)$$

with $\alpha_k = 2^{-k-1}\alpha_0$, $0 \le \alpha_0 < \pi$ has the limit function in $C^1(\mathbb{R})$. Moreover, if the initial control polygon is a regular n-gon and we choose $\alpha_0 = \frac{2\pi}{n}$, then the limit curve is the circle inscribed in that regular n-gon.

The following subdivision scheme is an interpolatory subdivision scheme for design of curves.

Examples 1.3. (Four Point Scheme) [27, 32] The scheme is defined by the mask

$$a := \{a_{-3} = a_3 = -w,\ a_{-1} = a_1 = \frac{1}{2} + w,\ a_0 = 1\}, a_i = 0,\ i \ne -3, 3, 0, -1, 1.$$

It has been known that the scheme converges for $|w| < 1/2$ and the limit function belongs to $C^1(\mathbb{R})$ for the range $0 < w < 0.154$ [28, 29]. Moreover the associated characteristic polynomial $a(z)$ can be written in factorized form as given below [29]:

$$a(z) = z^{-3}(1+z)^2(\frac{1}{2}z^2 - w(z-1)^2(1+z^2)).$$

The subdivision schemes for curves are easily generalized into schemes for surfaces with regular topology. In case, data set is prescribed on grids with irregular topologies uniform subdivision rule cannot be applied to all the faces and control points. Therefore it is of great importance to develop subdivision schemes applicable to such situations.

Doo-Sabin [26] and Catmull-Clark [16] (see Figure 1.1) first developed such schemes. We describe below the general procedure for these two schemes.

It is enough to consider the set of initial control points P^0 as vertices of a polyhedron which has a face of order n sorrounded by a layer of regular meshes. Let P^k denote the set of control points at kth level. Then the set P^{k+1} is obtained from the set P^k by a matrix operation [63]:

$$P^{k+1} = MP^k. \tag{1.12}$$

The matrix M is called the subdivision matrix associated with the scheme. In the analysis some results from the theory of circulant matrices are used.

Definition 1.4. (Circulant Matrix) [25] The block matrix

$$A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{pmatrix}$$

where A_i, $i = 0, 1, \ldots, n-1$ are matrices of order (m, m), is called a block circulant matrix of order (n, m) and is denoted by $A = bcirc(A_0, A_1, \cdots, A_{n-1})$.

Denoting by $u = w_n := exp(i\frac{2\pi}{n})$. Define the Fourier matrix F_n as

$$F_n^* = n^{-1/2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & u & u^2 & \cdots & u^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & u^{n-1} & u^{n-2} & \cdots & u \end{pmatrix}$$

where F^* is the conjugate transpose of a matrix F. It is known [25] that

$$(i)\, F_n = \bar{F}_n^*, \quad (ii) F_n^{-1} = F_n^*, \quad (iii)(F_n^*)^{-1} = F_n.$$

For $A = \text{bcirc}(A_0, A_1, \ldots, A_{n-1})$, the $m \times m$ matrices $\hat{A}_k$, $k = 0, 1, \ldots, n-1$ are defined by

$$\hat{A}_k = \sum_{j=0}^{n-1} w_n^{-jk} A_j. \tag{1.13}$$

The following result holds [25]

Theorem 1.4. (Diagonalization of a block circulant matrix) *A block circulant matrix $A = bcirc(A_0, A_1, \ldots, A_{n-1})$ has the diagonalization*

$$A = (F_n \times F_m)^* diag(F_m \hat{A}_0 F_m^*, \cdots, F_m \hat{A}_{n-1} F_m^*)(F_n \times F_m).$$

The following is a corollary of the above diagonalization result.

Corollary 1.5. [25] *Let λ be an eigenvalue of a particular matrix $\hat{A}_j$, then it is also an eigenvalue of $\hat{A}_{n-j}$ as well. Moreover, if $\hat{v}$ is an eigenvector of $\hat{A}_j$ corresponding to λ then the vector v defined by*

$$v := [\hat{v}, w^j \hat{v}, \cdot, w^{(n-1)} \hat{v}]^T \tag{1.14}$$

and $\bar{v}$, the complex conjugate of v are also eigenvectors of A with eigenvalue λ, where T denotes the block transpose.

It is known that in case of Doo-Sabin and Catmull-Clark schemes the matrix M of (1.12) is a block circulant matrix [63]. This class of schemes have been recently studied in a unified way by U. Reif [63]. He has shown that the Doo-Sabin class of subdivision schemes converges and the limit surface has a continuous tangent plane everywhere.

Briefly, one variant of Doo-Sabin scheme is described by the following. Let F be a face of P^0 with n vertices $A_1, \ldots, A_n$. Then the scheme defines the new vertex a_i corresponding to A_is by the subdivision rule:

$$a_i = \frac{4n+2}{8n} A_i + \frac{n+2}{8n}(A_{i+1} + A_{i-1}) + \frac{2}{8n}(A_{i+2} + \cdots + A_{i+n-1}).$$

A comprehensive collection of all available subdivision schemes for arbitrary topology and their analysis is contained in [73].

Chapter 2

Generalized Bezier curves

In this chapter a binary subdivision scheme for evaluation of the generalized Bernstein Bezier curves is presented. This scheme is a generalization of the Lane-Riesenfeld algorithm for polynomial Bezier curve. Moreover in this chapter we present a procedure to find the $\mathcal{D}$-convex hull of a subset of $\mathbb{R}^2$.

2.1 Generalized Bernstein-Bezier Representation

In this section we recall some basic concepts of the unified Bernstein-Bézier theory [38].

Let $\mathcal{D}$ be the two dimensional null space of a second order constant coefficient differential operator L of the form

$$L := D^2 + \gamma D + \delta, \quad \gamma, \delta \in \mathbb{R}. \tag{2.1}$$

Let $d \in \mathcal{D}$ be the unique solution of the initial value problem

$$Ld = 0, \quad d(0) = 0, \quad Dd(0) = 1. \tag{2.2}$$

It follows that $\mathcal{D} = \text{span}\{d(\cdot - t) \ : \ t \in \mathbb{R}\}$. Since $\mathcal{D}$ is two dimensional, for any two a and b such that $d(a-b) \neq 0$ (and therefore $d(b-a) \neq 0$) we have

$$\mathcal{D} = \text{span}\{d(\cdot - a), d(\cdot - b)\}.$$

For $n \geq 0$, let $\mathcal{D}_n := \text{span}\{d^n(\cdot - t) \ : \ t \in \mathbb{R}\}$. The elements of $\mathcal{D}_n$ are called $\mathcal{D}$-polynomials of degree n.

The following result is proved in [38].

Theorem 2.1. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and $d(b-a) \neq 0$. Then the functions*

$$B_i^n(x;a,b) := \binom{n}{i}\left(\frac{d(x-b)}{d(a-b)}\right)^{n-i}\left(\frac{d(x-a)}{d(b-a)}\right)^{i}, \quad x \in \mathbb{R},\ i = 0,1,\ldots,n \tag{2.3}$$

form a basis for $\mathcal{D}_n$.

We call the functions B_i^n, $i = 0,1,\cdots,n$ the Bernstein basis polynomials of degree n coressponding to $\mathcal{D}$ or simply $\mathcal{B}$-polynomials.

In case $\gamma = 0$ and $\delta = 0$, then $\mathcal{D}_n = \Pi_n$, where Π_n is the space of algebraic polynomials of degree n. On the other hand if $\gamma = 0$ and $\delta = 1$ then we have $\mathcal{D}_n = \mathcal{T}_n$, where

$$\mathcal{T}_n = \begin{cases} \text{span}\{1, sin(2x), cos(2x), sin(4x), cos(4x), \ldots, sin(nx), cos(nx)\}, \\ \qquad n \text{ even}, \\ \text{span}\{sin(x), cos(x), sin(3x), cos(3x), \ldots, sin(nx), cos(nx)\}, \\ \qquad n \text{ odd}, \end{cases} \tag{2.4}$$

is the usual space of *trigonometric polynomials of order* $n+1$ [1, 48]. Clearly when $\mathcal{D}_n = \Pi_n$ the $\mathcal{B}$-polynomials B_i^n, $i = 0,1,\ldots n$ are the classical Bernstein basis polynomials [36] and in case $\mathcal{D}_n = \mathcal{T}_n$, the $\mathcal{B}$-polynomials are the circular Bernstein basis polynomials [1], or trigonometric Bernstein basis polynomials [48].

Definition 2.1. Let $a, b \in \mathbb{R}$ such that $d(b-a) \neq 0$. Then for any $p_i \in \mathbb{R}$, $i = 0,1,\ldots,n$ we call

$$f(x) := \sum_{i=0}^{n} p_i\, B_i^n(x;a,b), \quad x \in \mathbb{R} \tag{2.5}$$

a generalized Bernstein-Bézier or GBB-polynomial of degree n.

The points $\{(\xi_i, c_i) : i = 0,1,\ldots n\}$, where

$$\xi_i = a + i\frac{b-a}{n}, \quad i = 0,1,\ldots,n, \tag{2.6}$$

are called *control points* of f.

Let us define $\mathcal{L}_n := \text{span}\{d(n \cdot -t) : t \in \mathbb{R}\}$.

Definition 2.2. (Control curve):
Let $f(x)$, $x \in [a,b]$ be defined as in (2.5). Let g_i, $i = 1,2,\ldots,n$, be the unique functions

from $\mathcal{L}_n$, which interpolate the control points (ξ_{i-1}, p_{i-1}), (ξ_i, p_i), $i = 1, 2, \ldots, n$. More precisely, define

$$g_i(t) := \frac{d(n(t-\xi_i))}{d(n(\xi_{i-1}-\xi_i))}\, p_{i-1} + \frac{d(n(t-\xi_{i-1}))}{d(n(\xi_i-\xi_{i-1}))}\, p_i. \tag{2.7}$$

We call the curve consisting of the pieces $G_i := \{(t, g_i(t)),\ t \in [\xi_{i-1}, \xi_i]\}$, $i = 1, \ldots, n$, the *control curve* of f.

Definition 2.3. ($\mathcal{D}$-convex set):
A subset B of $\mathbb{R}^2$ is called $\mathcal{D}$-convex of degree n, $n \geq 2$ if for any two points (t_1, c_1) and (t_2, c_2) of B such that $d(t_2 - t_1) \neq 0$, the curve

$$\Big\{\Big(t, \frac{d(n(x-t_2))}{d(n(t_1-t_2))}\, c_1 + \frac{d(n(x-t_1))}{d(n(t_2-t_1))}\, c_2\Big) : t \in [t_1, t_2]\Big\}$$

which is the $\mathcal{D}$-convex combination of degree n of (t_1, c_1) and (t_2, c_2) lies in B.

Definition 2.4. ($\mathcal{D}$-convex hull):
The D-convex hull of degree n of a subset B of $\mathbb{R}^2$ is the smallest $\mathcal{D}$-convex set of degree n containing B.

Trigonometric convex hull of degree n which is introduced in [1, 48] is a special case of $\mathcal{D}$-convex hull.

If $h(t)$ is a GBB-polynomial on $[a, b]$ with control points $\{(t_i, c_i),\ i = 0, 1, \ldots, n\}$ then it can be shown [38] that $\{(t, h(t)) : t \in [a, b])\}$ lies in the $\mathcal{D}$-convex hull of degree n of its control points $\{(t_i, c_i) : i = 0, 1, \ldots, n\}$.

2.2 Properties of $d(x)$

We now give some properties of $d(x)$ which will be useful in the subsequent sections.

We recall here that $d(x)$ is the solution of the initial value problem (2.2). Let $r^2 + r\gamma + \delta = 0$ be the characteristic equation of the differential equation $Lf = 0$, where L is the differential operator defined by (2.1). Let r_1 and r_2 be the roots of the characteristic equation. Depending upon the values of r_1 and r_2 we have the following cases:

(A) $d(x) = \frac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2} = 2\frac{e^{-\gamma x/2}\sinh((r_1 - r_2)x/2)}{r_1 - r_2}$, if r_1 and r_2 are real and distinct.

(B) $d(x) = x\, e^{r_1 x} = x\, e^{-\gamma x/2}$ if $r_1 = r_2 = \gamma$.

(C) $d(x) = e^{-\gamma x/2}\sin(\beta x)/\beta$ if $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$.

Note that $d(x) > 0$ for $x \in (0, z)$ where $z = \infty$ for the cases (A) and (B) and $z = \pi/\beta$ for the case (C). Moreover $d(x)$ can have at most one zero in any interval I of length $|I| < z$. Therefore the function $d(\cdot - t)$, $t \in \mathbb{R}$ can have at most one zero in any interval I of length $|I| < z$.

Observe that $d(nx)$ is the solution of the initial value problem:

$$L_n f = 0;\ f(0) = 0;\ f'(0) = n \tag{2.8}$$

where $L_n := D^2 + n\gamma D + \delta n^2$ and $\mathcal{L}_n$ is the null space of the operator L_n. Moreover $d(n\,\cdot)$ and its translations $d(n \cdot -t)$, $t \in \mathbb{R}$ can have at most one zero in any interval I of length $|I| < \frac{z}{n}$. This leads to the following lemmas which we use in the sequel.

Lemma 2.2. *Let $u(x) \in \mathcal{L}_n$. Then $u(x)$ can have at most one root in any interval I of length $|I| < z/n$.*

Proof: Let $x_0 \in \mathbb{R}$ be such that $u(x_0) = 0$ and $u'(x_0) = M$ for some $M \in \mathbb{R}$. Then $u(x)$ is the unique solution of the initial value problem:

$$L_n(f) = 0;\ f(x_0) = 0;\ f'(x_0) = M.$$

Hence $u(x) = \frac{M}{n} d(n(x - x_0))$. Therefore $u(x) \neq 0$ for $|n(x - x_0)| < z$ or $|x - x_0| < \frac{z}{n}$. Thus the lemma follows. $\square$

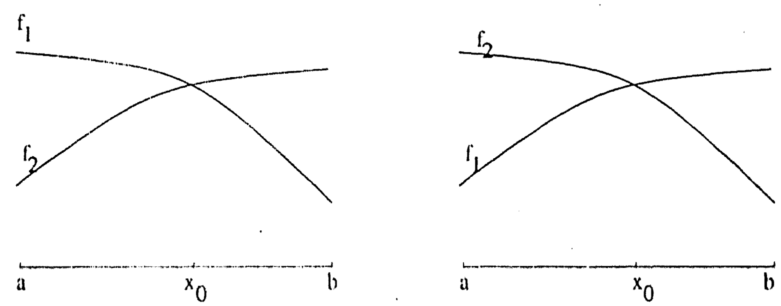


Figure 2.1: Two distinct solutions of $\mathcal{L}_n$ intersect each other only once.

Remark 2.3. Let $a, b \in \mathbb{R}$ such that $0 < b - a < z/n$. Let $f_1, f_2 \in \mathcal{L}_n$ such that $f_1 \neq f_2$. Let $f_1(x_0) = f_2(x_0)$ for some $x_0 \in [a, b]$. Then by Lemma 2.2 and the fact that $f_1'(x_0) \neq f_2'(x_0)$, we have either (see Figure 2.1)

$$f_1(x) < f_2(x),\ x \in [a, x_0) \quad \text{and} \quad f_1(x) > f_2(x),\ x \in (x_0, b]$$

or

$$f_1(x) > f_2(x),\ x \in [a,\ x_0) \quad \text{and} \quad f_1(x) < f_2(x),\ x \in (x_0,\ b].$$

Lemma 2.4. *Let $a, b \in \mathbb{R}$ and $0 < b - a < \frac{\pi}{\beta}$. Then the function*

$$g(x) := \frac{\sin(\beta(x-a))}{\sin(\beta(b-a))} + \frac{\sin(\beta(b-x))}{\sin(\beta(b-a))},\quad x \in [a,\ b]$$

has the maximum at $\frac{a+b}{2}$. Hence $\max\{g(x) : x \in [a,\ b]\} = \frac{1}{\cos(\beta\frac{b-a}{2})}$.

Proof: We first note that

$$g'(x) = \beta\left(\frac{\cos(\beta(x-a))}{\sin(\beta(b-a))} - \frac{\cos(\beta(b-x))}{\sin(\beta(b-a))}\right),\quad x \in [a,\ b]$$

is a solution of the differential operator $L := D^2 + \beta^2$. Therefore g' can have at most one zero in $[a, b]$. Since $g'(\frac{a+b}{2}) = 0$, $\frac{a+b}{2}$ is the only zero of g' in $[a,\ b]$. Noting that $g(a) = g(b) = 1$ and $\frac{\sin(\beta x)}{\beta x}$ is a strictly decreasing function in $(0,\ \frac{\pi}{\beta})$, we have

$$g(x) \geq \frac{x-a}{b-a} + \frac{b-x}{b-a} = 1 \quad \forall x \in (a, b).$$

Therefore, g has maximum equal to $\frac{1}{\cos(\beta\frac{b-a}{2})}$ at $x = \frac{a+b}{2}$. □

In an analogous manner, one proves using strictly increasing property of $\frac{\sinh(\beta x)}{\beta x}$ in $(0,\ \infty)$, the following lemma.

Lemma 2.5. *Let $a, b \in \mathbb{R}$ and $0 < b - a < \infty$. Then the function*

$$g(x) := \frac{\sinh(\beta(x-a))}{\sinh(\beta(b-a))} + \frac{\sinh(\beta(b-x))}{\sinh(\beta(b-a))},\quad x \in [a,\ b]$$

has the minimum at $\frac{a+b}{2}$. Hence $\min\{g(x) : x \in [a,\ b]\} = \frac{1}{\cosh(\beta\frac{b-a}{2})}$.

In the following proposition we give some properties of $d(x)$ which will be used in finding certain estimates in the next sections.

Proposition 2.6. *For $0 < l < z$, we have:*

(i) $\frac{d(-x/2)}{d(-x)} > 0$ *and* $\frac{d(x/2)}{d(x)} > 0 \quad \forall x \in (0, l]$.

(ii) Either $\frac{d(x/2)}{d(x)} \geq \frac{d(-x/2)}{d(-x)} \quad \forall x \in (0, l]$ *or* $\frac{d(x/2)}{d(x)} < \frac{d(-x/2)}{d(-x)} \quad \forall x \in (0, l]$.

(iii) Either $\frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)} > 1 \quad \forall x \in (0,\ l]$ *or* $\frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)} \leq 1 \quad \forall x \in (0, l]$.

(iv) There exists a constant C_1 depending only upon γ, δ and l such that for all $x \in [-l, l]$ we have $|d(x)| \leq C_1|x|$.

Proof: The values of $\frac{d(-x/2)}{d(-x)}$ for the cases (A), (B) and (C) are $\frac{e^{(r_1+r_2)x/2}}{e^{r_1x/2}+e^{r_2x/2}}$, $\frac{e^{r_1x/2}}{2}$ and $e^{\alpha x/2}\frac{\sin(\beta x/2)}{\sin(\beta x)}$ respectively. Similarly the values of $\frac{d(x/2)}{d(x)}$ for the case (A), (B) and (C) are $\frac{1}{e^{r_1x/2}+e^{r_2x/2}}$, $\frac{1}{2}e^{-r_1x/2}$ and $e^{-\alpha x/2}\frac{\sin(\beta x/2)}{\sin(\beta x)}$ respectively. This proves (i) and (ii).

The values of $\frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)}$ for the cases (A), (B) and (C) are $\frac{e^{(r_1+r_2)x/2}+1}{e^{r_1x/2}+e^{r_2x/2}}$, $\cosh(\gamma x/4)$ and $\frac{\cosh(\gamma x/4)}{\cos(\beta x/2)}$ respectively. This proves (iii).

The proof of (iv) follows from the mean value theorem for the function $d(x)$. $\square$

The following lemma is also needed in the sequel.

Lemma 2.7. *Let $0 < l < z$ and $0 < v - u \leq l$. Then there exists a constant C_4 depending only upon γ, δ and l such that for all $x \in [u, v]$*

$$h(x) := \left|\frac{d(x-u)}{d(v-u)} - \frac{x-u}{v-u}\right| + \left|\frac{d(x-v)}{d(u-v)} - \frac{x-v}{u-v}\right| \leq C_4(v-u).$$

Moreover, in case $\gamma = 0$

$$h(x) \leq C_4(v-u)^2, \quad x \in [u, v].$$

Proof: We prove the lemma by considering different cases ((B), (C) and (A) of this section) of $d(x)$. Let $h_1(x) = \left|\frac{d(x-u)}{d(v-u)} - \frac{x-u}{v-u}\right|$ and $h_2(x) = \left|\frac{d(x-v)}{d(u-v)} - \frac{x-v}{u-v}\right|$.

When $d(x) = x\,e^{-\gamma x/2}$ we have

$$h_1(x) \leq |e^{\gamma(v-x)/2} - 1| \quad \text{and} \quad h_2(x) \leq |e^{-\gamma(x-u)/2} - 1|.$$

Note that

$$|e^{\gamma(v-x)/2} - 1| + |e^{-\gamma(x-u)/2} - 1| \leq e^{|\gamma|(v-u)/2} - e^{-|\gamma|(v-u)/2}. \tag{2.9}$$

By applying the mean value theorem for the function $e^{|\gamma|x}$ on $[\frac{u-v}{2}, \frac{v-u}{2}]$ we get

$$|e^{\gamma(v-x)/2} - 1| + |e^{-\gamma(x-u)/2} - 1| \leq (v-u)|\gamma|\,e^{|\gamma|l/2}. \tag{2.10}$$

Therefore choose $C_4 = |\gamma|\,e^{|\gamma|l/2}$ in this case.

Suppose $d(x) = e^{-\gamma x/2}\sin(\beta x)/\beta$. Note that in this case $z = \pi/\beta$. Therefore

$$\begin{aligned} h_1(x) &= \left|e^{-\gamma(x-v)/2}\frac{\sin(\beta(x-u))}{\sin(\beta(v-u))} - \frac{x-u}{v-u}\right| \\ &\leq e^{-\gamma(x-v)/2}\left|\frac{\sin(\beta(x-u))}{\sin(\beta(v-u))} - \frac{x-u}{v-u}\right| + \frac{x-u}{v-u}|e^{-\gamma(x-v)/2} - 1|. \end{aligned} \tag{2.11}$$

Since $\frac{\sin(\beta x)}{\beta x}$ is a strictly decreasing function in $(0, z)$ we have

$$\frac{\sin(\beta(x-u))}{\sin(\beta(v-u))} > \frac{x-u}{v-u}.$$

Therefore

$$h_1(x) \le e^{|\gamma|(v-u)/2}\left\{\frac{\sin(\beta(x-u))}{\sin(\beta(v-u))} - \frac{x-u}{v-u}\right\} + |e^{-\gamma(x-v)/2} - 1|.$$

Similarly we have

$$h_2(x) \le e^{|\gamma|(v-u)/2}\left\{\frac{\sin(\beta(x-v))}{\sin(\beta(u-v))} - \frac{x-v}{u-v}\right\} + |e^{-\gamma(x-u)/2} - 1|.$$

Thus by (2.10) we get

$$h(x) \le e^{|\gamma|(v-u)/2}\left\{\frac{\sin(\beta(x-u))}{\sin(\beta(v-u))} + \frac{\sin(\beta(x-v))}{\sin(\beta(u-v))} - 1\right\} + (v-u)\,|\gamma|\,e^{|\gamma|l/2}.$$

By Lemma 2.4 we have

$$\begin{aligned} h(x) &\le e^{|\gamma|(v-u)/2}\left\{\frac{1}{\cos(\beta(v-u)/2)} - 1\right\} + (v-u)\,|\gamma|\,e^{|\gamma|l/2} \\ &\le e^{|\gamma|(v-u)/2}\left\{\frac{2\sin^2(\beta(v-u)/4)}{\cos(\beta\,\frac{v-u}{2})}\right\} + (v-u)\,|\gamma|\,e^{|\gamma|l/2} \end{aligned}$$

Choose $C_4 = \left(\frac{\beta^2 l}{8\cos(\beta l/2)} + |\gamma|\right) e^{|\gamma|l/2}$ in this case.

Similarly using the fact that $\sinh(x) \le x\,e^l$, one can prove that there exists a constant C_4 such that $h(x) \le C_4(v-u)$ in case $d(x) = 2\frac{e^{-\gamma x/2}}{r_1 - r_2}\sinh((r_1 - r_2)x/2)$.

In case $\gamma = 0$ we have

$$h(x) \le \max\{2\frac{\sin^2(\beta(v-u)/4)}{\cos(\beta\frac{v-u}{2})}, 2\frac{\sinh^2((r_1-r_2)(v-u)/4)}{\cos((r_1-r_2)\frac{v-u}{2})}\}$$

for all possible cases of $d(x)$. Therefore there exists C_4 such that $h(x) \le C_4(v-u)^2$. This proves the lemma. □

2.3 Subdivision Algorithm and its Refinement Equation

We define the GBB-curve as the function

$$F(x) = \sum_{i=0}^{n} P_i\, B_i^n(x; a, b), \quad x \in [a, b]$$

where $P_i \in \mathbb{R}^d$, $d > 1$. To generate the curve F it is enough to generate each of its components. Thus we now present the subdivision algorithm for the evaluation of GBB-polynomial

$$f(x) = \sum_{i=0}^{n} q_i^0 \, B_i^n(x; a, b), \quad x \in [a, b] \tag{2.12}$$

where $q_i^0 \in \mathbb{R}$, $i = 0, 1, \ldots, n$ and $0 < b - a < z$.

We denote $\frac{d(-\frac{l}{2})}{d(-l)}$ and $\frac{d(\frac{l}{2})}{d(l)}$ by $b_0(l)$ and $b_1(l)$ respectively.

Algorithm 2.1.
Given the control points q_i^0, $i = 0, 1, \ldots, n$, we obtain the control points q_i^1, $i = 0, 1, \ldots, 2n$ as follows:
Let $p_i^0 = q_i^0$, $i = 0, 1, \ldots, n$.
For $k = 1, \ldots, n$, set

$$p_i^k = b_0(l) \, p_{i-1}^{k-1} + b_1(l) \, p_i^{k-1}, \quad i = k, k+1, \ldots n. \tag{2.13}$$

Define

$$q_i^1 = \begin{cases} p_i^i, & i = 0, 1, \ldots, n, \\ p_n^{2n-i}, & i = n, n+1, \ldots, 2n. \end{cases} \tag{2.14}$$

In particular, $p_i^1 = b_0(l) \, p_{i-1}^0 + b_1(l) \, p_i^0$ and

$$\begin{aligned} p_i^2 &= b_0(l) \, p_{i-1}^1 + b_1(l) \, p_i^1 \\ &= (b_0(l))^2 \, p_{i-2}^0 + 2 \, b_0(l) \, b_1(l) \, p_{i-1}^0 + (b_1(l))^2 \, p_i^0. \end{aligned}$$

By induction we get

$$p_i^k = \sum_{j=0}^{k} \binom{k}{j} (b_0(l))^j \, (b_1(l))^{k-j} \, p_{i-j}^0, \quad k = 0, 1, \ldots, n, \; i = k, k+1, \ldots, n.$$

Therefore the set of control points q_i^1, $i = 0, 1 \ldots, 2n$ defined by (2.14) can be written explicitly as follows:

$$\begin{aligned} q_k^1 &= \sum_{s=0}^{k} q_s^0 \binom{k}{s} (b_0(l))^{k-s} \, (b_1(l))^s, \quad k = 0, 1, \ldots, n \\ q_{n+k}^1 &= \sum_{s=k}^{n} q_s^0 \binom{n-k}{n-s} (b_0(l))^{n-s} \, (b_1(l))^{s-k}, \quad k = 0, 1, \ldots, n. \end{aligned} \tag{2.15}$$

After the rth iteration, the control points $\{q_k^{r+1},\ k = 0, 1, \dots, 2^{r+1}\, n\}$ are obtained from $\{q_k^r,\ k = 0, 1, \dots, 2^r\, n\}$ as follows: For each $j = 0, 1, \dots, 2^r$ replace q_k^0 by q_{jn+k}^r, $k = 0, 1, \dots, n$ in the above algorithm. Then the resulting points in (2.14) are q_{2jn+k}^{r+1}, $k = 0, 1, \dots, 2n$. Subsequently, we get

$$\begin{aligned} q_{2jn+k}^{r+1} &= \sum_{s=0}^{k} q_{jn+s}^{r} \binom{k}{s} (b_0(\frac{l}{2^r}))^{k-s} (b_1(\frac{l}{2^r}))^{k}, \quad s = 0, 1, \dots, n \\ q_{2jn+n+k}^{r+1} &= \sum_{s=k}^{n} q_{jn+s}^{r} \binom{n-k}{n-s} (b_0(\frac{l}{2^r}))^{n-s} (b_1(\frac{l}{2^r}))^{s-k}, \quad s = 0, 1, \dots, n. \end{aligned} \tag{2.16}$$

In case $l = b - a$, the algorithm satisfies the following refinement equation:

$$\sum_{i=0}^{n} q_i^0\, B_i^n(x; a, b) = \begin{cases} \sum_{i=0}^{n} q_i^1\, B_i^n(x; a, \frac{a+b}{2}), & x \in [a, \frac{a+b}{2}], \\ \sum_{i=0}^{n} q_{n+i}^1\, B_i^n(x; \frac{a+b}{2}, b), & x \in [\frac{a+b}{2}, b] \end{cases} \tag{2.17}$$

which is immediate from the following theorem.

Theorem 2.8. *Let $a, b \in \mathbb{R}$ be such that $0 < l = b - a < z$ and $x' = \frac{a+b}{2}$. Then for $p_i^0 \in \mathbb{R}$, $i = 0, 1, \dots, n$ we have*

$$\sum_{i=0}^{n} p_i^0\, B_i^n(x; a, b) = \begin{cases} \sum_{i=0}^{n} p_i^i\, B_i^n(x; a, x'), & x \in [a, x'], \\ \sum_{i=0}^{n} p_n^{n-i}\, B_i^n(x; x', b), & x \in [x', b], \end{cases} \tag{2.18}$$

where p_i^k, $k = 1, 2, \dots, n$; $i = k, k+1, \dots, n$ are defined by (2.13).

Proof: We prove the theorem by induction on n. When $n = 1$,

$$\sum_{i=0}^{1} p_i^0\, B_i^1(x; a, b) = p_0^0\, \frac{d(x-b)}{d(a-b)} + p_1^0\, \frac{d(x-a)}{d(b-a)}. \tag{2.19}$$

Since

$$d(x-b) = d(\frac{a-b}{2})\, \frac{d(x-a)}{d(x'-a)} + d(a-b)\, \frac{d(x-x')}{d(a-x')}, \tag{2.20}$$

for $x \in [a, x']$ we get

$$\begin{aligned} \sum_{i=0}^{1} p_i^0\, B_i^1(x; a, b) &= \frac{p_0^0}{d(a-b)} \left\{ d(\frac{a-b}{2}) \frac{d(x-a)}{d(x'-a)} + d(a-b)\, \frac{d(x-x')}{d(a-x')} \right\} + p_1^0 \frac{d(x-a)}{d(b-a)} \\ &= \left(\frac{d(\frac{a-b}{2})}{d(a-b)}\, p_0^0 + \frac{d(\frac{b-a}{2})}{d(b-a)}\, p_1^0 \right) \frac{d(x-a)}{d(\frac{b-a}{2})} + p_0^0\, \frac{d(x-x')}{d(\frac{a-b}{2})} \end{aligned}$$

$$= p_0^0 \frac{d(x-x')}{d(\frac{a-b}{2})} + p_1^1 \frac{d(x-a)}{d(\frac{b-a}{2})}$$

$$= \sum_{i=0}^{1} p_i^i B_i^n(x; a, x')$$

In a similar way by expressing $d(x-a)$ in terms of $\frac{d(x-x')}{d(b-x')}$ and $\frac{d(x-b)}{d(x'-b)}$, we get

$$\sum_{i=0}^{1} p_i^0 B_i^1(x; a, b) = \sum_{i=0}^{1} p_1^{1-i} B_i^n(x; x', b), \quad x \in [x', b].$$

This proves (2.18) for the case $n = 1$.

Let us assume that (2.18) holds for $n = m-1$. We prove that (2.18) holds for the case $n = m$. By recurrence relation of $\mathcal{B}$-polynomials [38] we have for $x \in [a, x']$

$$\sum_{i=0}^{m} p_i^0 B_i^m(x; a, b) = \frac{d(x-b)}{d(-l)} \sum_{i=0}^{m-1} p_i^0 B_i^{m-1}(x; a, b) + \frac{d(x-a)}{d(l)} \sum_{i=0}^{m-1} p_{i+1}^0 B_i^{m-1}(x; a, b).$$

By induction assumption we get

$$\sum_{i=0}^{m} p_i^0 B_i^m(x; a, b) = \frac{d(x-b)}{d(-l)} \sum_{i=0}^{m-1} p_i^i B_i^{m-1}(x; a, x') + \frac{d(x-a)}{d(l)} \sum_{i=0}^{m-1} p_{i+1}^i B_i^{m-1}(x; a, x')$$

By (2.20) we get

$$\sum_{i=0}^{m} p_i^0 B_i^m(x; a, b)$$

$$= \frac{1}{d(a-b)} \left\{ d(\frac{a-b}{2}) \frac{d(x-a)}{d(\frac{b-a}{2})} + d(a-b) \frac{d(x-x')}{d(\frac{a-b}{2})} \right\} \sum_{i=0}^{m-1} p_i^i B_i^{m-1}(x; a, x')$$

$$+ \frac{d(x-a)}{d(\frac{b-a}{2})} \frac{d(\frac{b-a}{2})}{d(b-a)} \sum_{i=0}^{m-1} p_{i+1}^i B_i^{m-1}(x; a, x')$$

Rearranging the above equation we get

$$\sum_{i=0}^{m} p_i^0 B_i^m(x; a, b) = \frac{d(x-x')}{d(\frac{-l}{2})} \sum_{i=0}^{m-1} p_i^i B_i^{m-1}(x; a, x')$$

$$+ \frac{d(x-a)}{d(\frac{l}{2})} \sum_{i=0}^{m-1} \left(\frac{d(-\frac{l}{2})}{d(-l)} p_i^i + \frac{d(\frac{l}{2})}{d(l)} p_{i+1}^i \right) B_i^{m-1}(x; a, x')$$

$$= \frac{d(x-x')}{d(\frac{-l}{2})} \sum_{i=0}^{m-1} p_i^i B_i^{m-1}(x; a, x') + \frac{d(x-a)}{d(\frac{l}{2})} \sum_{i=0}^{m-1} p_{i+1}^{i+1} B_i^{m-1}(x; a, x')$$

$$= \sum_{i=0}^{m} p_i^i B_i^m(x; a, x').$$

Similarly, by expressing $d(x-a)$ in terms of $\frac{d(x-x')}{d(b-x')}$ and $\frac{d(x-b)}{d(x'-b)}$ we get

$$\sum_{i=0}^{m} p_i^0 \, B_i^m(x; a, b) = \sum_{i=0}^{m} p_m^{m-i} \, B_i^m(x; x', b), \quad x \in [x', b].$$

Hence the theorem follows by induction. □

Remark 2.9. The above refinement equation can also be proved using polar forms of a GBB function (see Remark 4.7(c) of [38]).

After k iterations we have the following set of equations: for $r = 0, 1, \ldots, 2^k - 1$

$$\sum_{i=0}^{n} q_i^0 \, B_i^n(x; a, b) = \sum_{i=0}^{n} q_{rn+i}^k \, B_i^n(x; a_r^k, a_{r+1}^k), \quad x \in [a_r^k, a_{r+1}^k], \tag{2.21}$$

where $a_r^k = a + r\frac{b-a}{2^k}$.

Note that the function

$$s_{k,r}(x) := \sum_{i=0}^{n} q_{rn+i}^k \, B_i^n(x; a_r^k, \; a_{r+1}^k), \quad x \in [a_r^k, a_{r+1}^k] \tag{2.22}$$

is itself a GBB function defined on $[a_r^k, a_{r+1}^k]$ and has the control points

$$\{(\xi_{i,r}^k, \; q_{rn+i}^k) : \quad i = 0, 1, \ldots, n\}$$

where

$$\xi_{i,r}^k = a_r^k + i\frac{a_{r+1}^k - a_r^k}{n}, \quad r = 0, 1, \ldots, 2^k - 1, \; i = 0, 1, \ldots, n. \tag{2.23}$$

For $0 < |\xi_{i,r}^k - \xi_{j,r}^k| < \frac{z}{n}$, let us define $f_{i,j}^{r,k}(x)$ be the function in $\mathcal{L}_n$ which interpolates q_{rn+i}^k and q_{rn+j}^k at $\xi_{i,r}^k$ and $\xi_{j,r}^k$ respectively i.e.,

$$f_{i,j}^{r,k}(x) := \frac{d(n(x - \xi_{j,r}^k))}{d(n(\xi_{i,r}^k - \xi_{j,r}^k))} \, q_{rn+i}^k + \frac{d(n(x - \xi_{i,r}^k))}{d(n(\xi_{j,r}^k - \xi_{i,r}^k))} \, q_{rn+j}^k. \tag{2.24}$$

Let $f^k(x)$, $x \in [a, b]$ be the function such that

$$f^k(x)\Big|_{[\xi_{i,r}^k, \xi_{i+1,r}^k]} = f_{i,i+1}^{r,k}(x). \tag{2.25}$$

Then the curve $\{(x, \; f^k(x)) : x \in [a_r^k, \; a_{r+1}^k]\}$ is the control curve for the GBB funct $s_{k,r}$. Both the curves $\{(x, \; f^k(x)) : \; x \in [a_r^k, \; a_{r+1}^k]\}$ and $\{(x, s_{k,r}(x)) : \; x \in [a_r^k, \; a_{r+}^k$ lie in the $\mathcal{D}$-convex hull of degree n of the corresponding control points

$$\{(\xi_{i,r}^k, \; q_{rn+i}^k) : \; i = 0, \ldots, n\}.$$

Moreover $s_{k,r}(a_r^k) = q_{rn}^k$ and $s_{k,r}(a_{r+1}^k) = q_{(r+1)n}^k$.

In Section 2.6 we show that the sequence of functions $\{f^k\}$ converges to the function $f(x)$ on $[a, b]$.

2.4 $\mathcal{D}$-Convex Hull

Let $t_0 < t_1 < \cdots < t_m$, $t_m - t_0 < \frac{z}{n}$, $b_i \in \mathbb{R}$, $i = 0, 1, \ldots, m$ and

$$B := \{(t_i, b_i) : i = 0, 1, \ldots, m\}.$$

In this section we show that the $\mathcal{D}$-convex hull of the set B is the region enclosed by a simple closed curve. This curve consists of graphs of functions from $\mathcal{L}_n$ interpolating some elements of B.

For any two points $(t_i,\ b_i)$, $(t_j,\ b_j)$ of B we define an $\mathcal{L}_n$-function by

$$g_{i,j}(x) := \frac{d(n(x - t_j))}{d(n(t_i - t_j))}\, b_i + \frac{d(n(x - t_i))}{d(n(t_j - t_i))}\, b_j, \quad x \in [t_0, t_m] \tag{2.26}$$

and the corresponding $\mathcal{L}_n$-curve:

$$G_{i,j} := \{(x,\ g_{i,j}(x)) \ : x \in [t_i, t_j]\}. \tag{2.27}$$

Note that $g_{i,j} \in \mathcal{L}_n$ and interpolates the points (t_i, b_i) and (t_j, b_j). Below we always assume $i < j$ in the definition of $g_{i,j}$.

Definition 2.5 (Edge curves). *Let $0 \le r < s \le m$. A curve $G_{r,s}$ is called an upper edge curve (UEC) of B if*

$$\begin{aligned} & g_{r,s}(t_i) \ge b_i \quad \forall t_i \in [t_r, t_s] \\ \textit{and} \quad & g_{r,s}(t_i) > b_i \quad \forall t_i \notin [t_r, t_s] \end{aligned} \tag{2.28}$$

We call $G_{r,s}$ a lower edge curve (LEC) of B if

$$\begin{aligned} & g_{r,s}(t_i) \le b_i \quad \forall t_i \in [t_r, t_s] \\ \textit{and} \quad & g_{r,s}(t_i) < b_i \quad \forall t_i \notin [t_r, t_s]. \end{aligned} \tag{2.29}$$

We show in this section that the $\mathcal{D}$-convex hull of B is the region enclosed by upper edge and lower edge curves.

Lemma 2.10. *There exist continuous curves U and L defined on $[t_0, t_m]$ which are union of UECs and LECs respectively. Moreover union of U and L is a simple closed curve.*

Proof: First of all we observe that there exists $u_1 \in \{1, 2, \ldots, m\}$ such that G_{0,u_1} is an UEC.

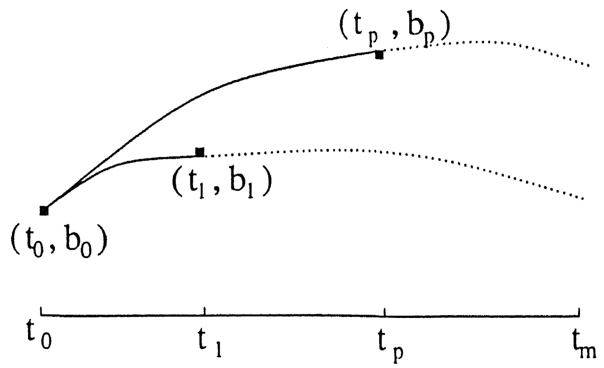


Figure 2.2: Existence of upper and lower edge curves

Indeed, if the function $g_{0,1}$ satisfies $g_{0,1}(t_i) > b_i$, $i = 2, 3, \ldots, m$ then $G_{0,1}$ is an UEC. Otherwise, there is a least $p \in \{2, 3, \ldots, m\}$ (see Figure 2.2) such that $b_p \geq g_{0,1}(t_p)$.

If $g_{0,1}(t_p) = b_p$ then by Remark 2.3 $g_{0,1} \equiv g_{0,p}$ and hence

$$g_{0,p}(t_i) > b_i,\ i \in \{2, 3, \ldots, p-1\} \quad \text{and} \quad g_{0,p}(t_i) = b_i,\ i = 0, 1, p.$$

On the other hand if $g_{0,1}(t_p) < b_p$ then $g_{0,1}$ and $g_{0,p}$ are distinct functions. Hence by Remark 2.3

$$g_{0,p}(t_i) > g_{0,1}(t_i) > b_i,\ i \in \{1, 2, \ldots, p-1\}.$$

In any case we get $g_{0,p}(t_i) \geq b_i$, $i \in \{0, 1, \ldots, p\}$. Now if $g_{0,p}(t_i) > b_i$ for all $i = p+1, \ldots, m$, $G_{0,p}$ is an UEC. Else there is a least $p_1 \in \{p+1, \ldots, m\}$ such that $g_{0,p}(t_{p_1}) \leq b_{p_1}$. Proceeding as before, we have

$$g_{0,p_1}(t_i) \geq b_i,\ i \in \{0, 1, \ldots, p_1\}.$$

Therefore in finite steps, we obtain some $u_1 \leq m$ such that G_{0,u_1} is an UEC.

In case $u_1 = m$, $U = G_{0,m}$. On the other hand if $u_1 < m$, proceeding as before and starting from u_1 instead of at 0 we find $u_2 \in \{u_1 + 1, \ldots, m\}$ such that

$$g_{u_1,u_2}(t_i) \geq b_i,\ i \in \{u_1, \ldots, u_2\} \quad \text{and} \quad g_{u_1,u_2}(t_i) > b_i,\ i \in \{u_2 + 1, \ldots, m\}.$$

As G_{0,u_1} is an UEC, we have $g_{0,u_1}(t_{u_2}) > b_{u_2} = g_{u_1,u_2}(t_{u_2})$. However in view of Remark 2.3, $g_{u_1,u_2}(t_i) > g_{0,u_1}(t_i) \geq b_i$, $i \in \{0, 1, \ldots, u_1\}$. This implies that G_{u_1,u_2} is an UEC.

Now if $u_2 = m$, $U = G_{0,u_1} \cup G_{u_1,u_2}$. Otherwise we repeat the above procedure to find the upper edge curve G_{u_2,u_3}. This procedure can be repeated to obtain in finite steps the UECs $G_{0,u_1}, G_{u_1,u_2}, \ldots, G_{u_r,m}$ which constitute U,

$$U = G_{0,u_1} \cup G_{u_1,u_2} \cup \ldots \cup G_{u_r,m}.$$

An analogous argument establishes that

$$L = G_{0,l_1} \cup G_{l_1,l_2} \cup \ldots \cup G_{l_s,m}$$

where $G_{0,l_1}, G_{l_1,l_2}, \ldots, G_{l_s,m}$ are LECs.

It remains to show that $U \cup L$ is a simple closed curve.

Clearly $U(x_0) = L(x_0)$ and $U(x_m) = L(x_m)$. Let $x_0 \in [t_p, t_{p+1}]$, $0 \leq p \leq m-1$ and $x_0 \neq t_0, t_m$. Then there exist an UEC $G_{u_i,u_{i+1}}$ and a LEC $G_{l_j,l_{j+1}}$ such that $x_0 \in [t_{u_i}, t_{u_{i+1}}]$ and $x_0 \in [t_{l_j}, t_{l_{j+1}}]$. Since $G_{u_i,u_{i+1}}$ is an UEC and $G_{l_j,l_{j+1}}$ is a LEC we have $g_{u_i,u_{i+1}}(t_p) \geq b_p$, $g_{u_i,u_{i+1}}(t_{p+1}) \geq b_{p+1}$, $g_{l_j,l_{j+1}}(t_p) \leq b_p$ and $g_{l_j,l_{j+1}}(t_{p+1}) \leq b_{p+1}$, and hence by Remark 2.3 we get $g_{l_j,l_{j+1}}(x_0) < g_{u_i,u_{i+1}}(x_0)$. This proves the lemma. □

For $x \in [t_0, t_m]$, let $u(x)$ and $l(x)$ be such that $(x, u(x)) \in U$ and $(x, l(x)) \in L$ respectively. Let us define

$$C := \{(x, y) : l(x) \leq y \leq u(x),\ x \in [t_0, t_m]\}.$$

Theorem 2.11. *C is the smallest $\mathcal{D}$-convex set of order n containing B.*

Proof: First we show that C is a $\mathcal{D}$-convex set of order n containing B.

For (x_0, y_0), $(x_1, y_1) \in C$ define $p(x)$:

$$p(x) = \frac{d(n(x-x_1))}{d(n(x_0-x_1))}\, y_0 + \frac{d(n(x-x_0))}{d(n(x_1-x_0))}\, y_1, \quad x \in [t_0, t_m].$$

Then we need to show that $l(x) \leq p(x) \leq u(x)$ for all $x \in [x_0, x_1]$. We show that $p(x) \leq u(x)$ and note that $l(x) \leq p(x)$ can be proved analogously.

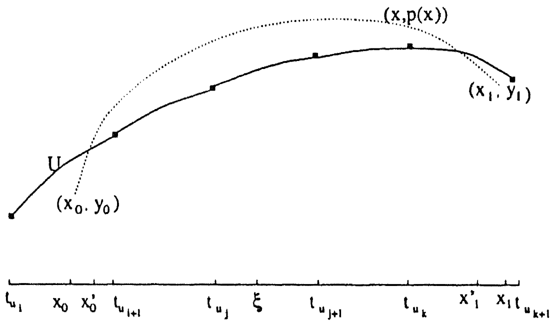


Figure 2.3: The case $p(\xi) > u(\xi)$

Suppose on the contrary there exists $\xi \in (x_0, x_1)$ such that $p(\xi) > u(\xi)$ and hence there exist $x'_0 \in (x_0, \xi)$ and $x'_1 \in (\xi, x_1)$ such that $p(x'_0) = u(x'_0)$ and $p(x'_1) = u(x'_1)$ (see Figure 2.3). Let us assume $(x'_0, u(x'_0)) \in G_{u_i,u_{i+1}}$, $(\xi, u(\xi)) \in G_{u_j,u_{j+1}}$ and $(x'_1, u(x'_1)) \in G_{u_k,u_{k+1}}$ for some i, j and k.

In case $x'_0, x'_1 \in [t_{u_j}, t_{u_{j+1}}]$, then by Remark 2.3, $p \equiv g_{u_j,u_{j+1}}$, which contradicts $p(\xi) > u(\xi)$.

Now suppose $x'_1 > t_{u_{j+1}}$. Since $G_{u_j,u_{j+1}}$ is an UEC, by using Remark 2.3 we get $g_{u_j,u_{j+1}}(x) > g_{u_k,u_{k+1}}(x)$ for all $x \in [t_{u_k}, t_m]$ if $t_{u_k} \neq t_{u_{j+1}}$ and for all $x \in (t_{u_k}, t_m]$ if $t_{u_k} = t_{u_{j+1}}$. Therefore $g_{u_j,u_{j+1}}(x'_1) > g_{u_k,u_{k+1}}(x'_1) = p(x'_1)$. Besides $p(\xi) > g_{u_j,u_{j+1}}(\xi)$ and hence p and $g_{u_j,u_{j+1}}$ intersect in the interval (ξ, x'_1). Similarly we can show that $g_{u_i,u_{i+1}}(x'_0) = p(x'_0) \leq g_{u_j,u_{j+1}}(x'_0)$. Therefore p and $g_{u_j,u_{j+1}}$ have a point of intersection in $[x'_0, \xi)$. Hence p and $g_{u_j,u_{j+1}}$ have two points of intersection in $[x'_0, x_1]$. This leads to a contradiction.

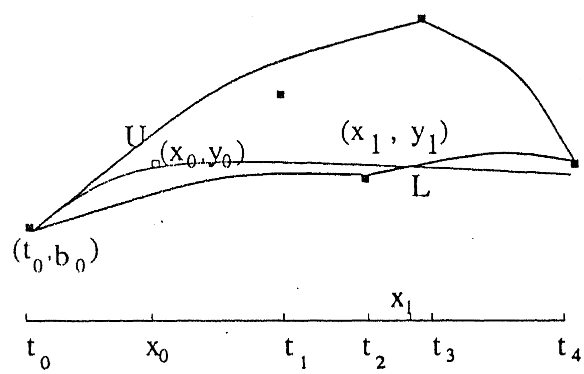


Figure 2.4: A typical $\mathcal{D}$-convex hull. Filled squares represent the control points

Now to see that C is the smallest $\mathcal{D}$-convex set of order n, let us take any $\mathcal{D}$-convex

set C_1 of order n containing B and show that every element of C is also an element of C_1. Indeed, let $(x_0, y_0) \in C$, then the curve

$$g(x) := \frac{d(n(x-x_0))}{d(n(t_0-x_0))}\, b_0 + \frac{d(n(x-t_0))}{d(n(x_0-t_0))}\, y_0, \quad x \in [t_0, t_m]$$

intersect at least one of the edge curves U or L, say at (x_1, y_1) which belongs to C_1 (see Figure 2.4). Since C_1 is a $\mathcal{D}$-convex set, $(x_0, y_0) \in C_1$ as well. This completes the proof of the theorem. □

Define for $r = 0, 1, \ldots, 2^k - 1$,

$$D_k(r) := \max_{\substack{x \in [\xi^k_{i,r}, \xi^k_{j,r}],\ y \in [\xi^k_{u,r}, \xi^k_{v,r}] \\ 0 \le i < j \le n,\ \xi^k_{j,r} - \xi^k_{i,r} < \frac{z}{n} \\ 0 \le u < v \le n,\ \xi^k_{v,r} - \xi^k_{u,r} < \frac{z}{n}}} \left\| (x, f^{r,k}_{i,j}(x)) - (y, f^{r,k}_{u,v}(y)) \right\| \tag{2.30}$$

where $f^{r,k}_{i,j}$ is defined in (2.24). Note that for a given n and for any $k > \log_2(\frac{nl}{z})$ the inequality $|\xi^k_{j,r} - \xi^k_{i,r}| < \frac{z}{n}$ is automatically satisfied for $r = 0, 1, \ldots, 2^k - 1$ and $0 \le i < j \le n$. The following theorem is an immediate consequence of Theorem 2.11.

Theorem 2.12. *When $k > \log_2(nl/z)$ and $r = 0, 1, \ldots, 2^k - 1$ the diameter of the $\mathcal{D}$-convex hull of order n of the set*

$$\{(\xi^k_{i,r}, q^k_{rn+i}) : i = 0, 1, \ldots, n\}$$

is equal to $D_k(r)$.

2.5 Convergence of Control Arcs and Control Polygons

In this section we study the convergence of $\{D_k(r)\}$ and show as a consequence the convergence of the control polygons and control arcs.

First of all, we show that for any $\epsilon > 0$, there exists an N such that $D_k(r) < \epsilon$ for $k \ge N$ and for $r = 0, 1, \ldots, 2^k - 1$. Towards that end we denote

$$Q(k) := \max_{0 \le i \le 2^k n} |q^k_i| \tag{2.31}$$

$$S(k) := \max_{0 \le i \le 2^k n - 1} |q_{i+1}^k - q_i^k| \tag{2.32}$$

and obtain some estimates for $Q(k)$ as well as $S(k)$.

Note that by Theorem 2.12, the D-convex hull H of order n of $\{(\xi_i^0, q_i^0)\}_{i=0}^n$ is a bounded subset of $\mathbb{R}^2$. Since all the k-th level control points

$$\{(\xi_{i,r}^k, q_{rn+i}^k) : i = 0, 1, \ldots, n, \ r = 0, 1, \ldots, 2^k - 1\}$$

and their intermediate points lie in H, there exists a constant C_q independent of i and j, and the level k such that

$$|p_i^j| \le C_q \tag{2.33}$$

and therefore, $Q(k) \le C_q \, \forall k$.

For $0 \le x < z$ define

$$\eta(x) := \begin{cases} 1, & \text{if } \frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)} \le 1, \\ 2\,\frac{d(x/2)}{d(x)}, & \text{if } \frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)} > 1 \text{ and } \frac{d(x/2)}{d(x)} \ge \frac{d(-x/2)}{d(-x)}, \\ 2\,\frac{d(-x/2)}{d(-x)}, & \text{if } \frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)} > 1 \text{ and } \frac{d(x/2)}{d(x)} < \frac{d(-x/2)}{d(-x)}. \end{cases} \tag{2.34}$$

Note that $\eta(x) \ge 1$ and $\frac{d(-x/2)}{d(-x)} + \frac{d(x/2)}{d(x)} \le \eta(x)$ for all $x \in [0, z)$.

For a fixed l, $0 \le l < z$, let us define $\sigma(0) = 1$ and for $j \ge 1$

$$\sigma(j) := \eta(l)\eta(\frac{l}{2}) \cdots \eta(\frac{l}{2^{j-1}}). \tag{2.35}$$

Observe that $\sigma(j+1) = \sigma(j)\eta(\frac{l}{2^j}) \; \forall j \ge 0$.

Lemma 2.13. *For a fixed l, $0 \le l < z$, there exists a constant C_σ such that*

$$\sigma(j) \le C_\sigma \quad \forall j \ge 0.$$

Proof: Suppose $\frac{d(-l/2)}{d(-l)} + \frac{d(l/2)}{d(l)} \le 1$. Then $\eta(l) = 1$. Moreover, by (iii) of Proposition 2.6 we have $\eta(\frac{l}{2^j}) = 1$. Therefore $\sigma(j) = 1$, $\forall j \ge 0$.

In case $\frac{d(-l/2)}{d(-l)} + \frac{d(l/2)}{d(l)} > 1$ and $\frac{d(l/2)}{d(l)} \ge \frac{d(-l/2)}{d(-l)}$ then $\eta(l) = 2\frac{d(l/2)}{d(l)}$, and by (ii) and (iii) of Proposition 2.6 we have $\eta(\frac{l}{2^j}) = 2\frac{d(\frac{l}{2^{j+1}})}{d(\frac{l}{2^j})} \; \forall j \ge 1$. This implies that

$$\sigma(j) = 2^j \frac{d(\frac{l}{2^{j+1}})}{d(l)} \quad \forall j \ge 0.$$

Therefore by (iv) of Proposition 2.6 there exists a constant C_1 such that $\sigma(j) \leq C_1 \frac{l}{d(l)}$, $\forall j \geq 0$.

Similarly we get $\sigma(j) \leq C_1 \frac{l}{|d(-l)|}$, $\forall j \geq 0$ for the case $\frac{d(-l/2)}{d(-l)} + \frac{d(l/2)}{d(l)} > 1$ and $\frac{d(l/2)}{d(l)} < \frac{d(-l/2)}{d(-l)}$. This proves the lemma. □

For $0 < x \leq l$ let us define

$$\alpha(x) = \max\Big\{ \,|2\frac{d(-x/2)}{d(-x)} - 1|, |2\frac{d(x/2)}{d(x)} - 1|\, \Big\}. \tag{2.36}$$

Theorem 2.14. *For $k \geq 1$*

$$S(k) \leq (\sigma(k))^{n-1}\Big\{C_q \sum_{j=0}^{k-1} \frac{\alpha(\frac{l}{2^j})}{2^{k-j-1}} + \frac{S(0)}{2^k}\Big\}. \tag{2.37}$$

Hence there exists a constant C_s such that for all k

$$S(k) \leq \begin{cases} \frac{C_s\, k}{2^k}, & \text{for general } \gamma \\ \frac{C_s}{2^k}, & \text{in case } \gamma = 0. \end{cases}$$

Proof: We prove the theorem by induction on k. By (2.14) we have

$$S(1) = \max_{0 \leq i \leq 2n-1} |q^1_{i+1} - q^1_i| = \max_{0 \leq i \leq n-1} \{|p^{i+1}_{i+1} - p^i_i|, |p^{n-i-1}_n - p^{n-i}_n|\}.$$

By (2.13) for a fixed i, $0 \leq i \leq n-1$ we have

$$\begin{aligned} |p^{i+1}_{i+1} - p^i_i| &= |\frac{d(-l/2)}{d(-l)}p^i_i + \frac{d(l/2)}{d(l)}p^i_{i+1} - p^i_i| \\ &\leq |\frac{d(-l/2)}{d(-l)}p^i_i - \frac{p^i_i}{2}| + |\frac{d(l/2)}{d(l)}p^i_{i+1} - \frac{p^i_{i+1}}{2}| + \frac{|p^i_{i+1} - p^i_i|}{2} \\ &\leq \alpha(l)\, C_q + \max_{i \leq j \leq n-1} \frac{|p^i_{j+1} - p^i_j|}{2}. \end{aligned}$$

For a fixed i and $j = i, i+1, \ldots, n-1$ we have

$$|p^i_{j+1} - p^i_j| \leq \Big(\frac{d(-l/2)}{d(-l)} + \frac{d(l/2)}{d(l)}\Big) \max_{i-1 \leq m \leq n-1} |p^{i-1}_{m+1} - p^{i-1}_m|.$$

Since, $(\frac{d(-l/2)}{d(-l)} + \frac{d(l/2)}{d(l)}) \leq \eta(l)$ and $\eta(l) > 1$ by repeating backwards we get

$$|p^i_{j+1} - p^i_j| \leq (\eta(l))^{n-1} S(0).$$

Therefore

$$\max_{0\le i\le n-1} |p_{i+1}^{i+1} - p_i^i| \le C_q\,\alpha(l) + (\eta(l))^{n-1}\frac{S(0)}{2}$$

and

$$\max_{0\le i\le n-1} |p_n^{n-i-1} - p_n^{n-i}| \le C_q\alpha(l) + (\eta(l))^{n-1}\frac{S(0)}{2}.$$

Since $\sigma(1) = \eta(l) > 1$ we get,

$$S(1) \le (\sigma(1))^{n-1}\{C_q\,\alpha(l) + \frac{S(0)}{2}\}.$$

This proves (2.37) for $k = 1$. Now, let us assume that (2.37) holds for $k = m$. Therefore,

$$S(m) \le (\sigma(m))^{n-1}\{C_q\sum_{j=0}^{m-1}\frac{\alpha(\frac{l}{2^j})}{2^{m-j-1}} + \frac{S(0)}{2^m}\}. \tag{2.38}$$

Recall that $\{q_i^{m+1}; i = 0, 1, \ldots, 2^{m+1}n\}$ is obtained from $\{q_i^m; i = 0, 1, \ldots, 2^m n\}$ by the subdivision algorithm by replacing l by $\frac{l}{2^m}$. Therefore, repeating arguments used for proving (2.37) for $k = 1$, we have

$$\begin{aligned} S(m+1) &\le (\eta(\frac{l}{2^m}))^{n-1}\{C_q\alpha(\frac{l}{2^m}) + S(m)/2\} \\ &\le (\eta(\frac{l}{2^m}))^{n-1}C_q\alpha(\frac{l}{2^m}) + (\sigma(m+1))^{n-1}\frac{C_q}{2}\sum_{j=0}^{m-1}\frac{\alpha(\frac{l}{2^j})}{2^{m-j-1}} + (\sigma(m+1))^{n-1}\frac{S(0)}{2^{m+1}}. \end{aligned}$$

Since, $1 \le \eta(\frac{l}{2^m}) \le \sigma(m+1)$ we have,

$$S(m+1) \le (\sigma(m+1))^{n-1}\{C_q\sum_{j=0}^{m}\frac{\alpha(\frac{l}{2^j})}{2^{m-j}} + \frac{S(0)}{2^{m+1}}\}.$$

This completes the induction and hence (2.37) is true for all k. By Lemma 2.7 and the fact that $\sigma(k) \le C_\sigma$ we have,

$$S(k) \le C_\sigma\{C_q\frac{2lm}{2^m} + \frac{S(0)}{2^m}\}, \quad \text{for any } \gamma$$

and

$$S(k) \le C_\sigma\{C_q\frac{4l}{2^m} + \frac{S(0)}{2^m}\}, \quad \text{for } \gamma = 0.$$

This completes the proof of the theorem. □

Our aim is to show that for every $\epsilon > 0$, there exists an N such that for every $r \in \{0, 1, \ldots, 2^k - 1\}$, $D_k(r) < \epsilon$ for $k \geq N$. Towards that end, we introduce the notion of kth level control polygon as follows: For any two points $(\xi^k_{i,r}, q^k_{rn+i})$ and $(\xi^k_{j,r}, q^k_{rn+j})$, we define

$$h^{r,k}_{i,j}(x) := \frac{\xi^k_{j,r} - x}{\xi^k_{j,r} - \xi^k_{i,r}} q^k_{rn+i} + \frac{x - \xi^k_{i,r}}{\xi^k_{j,r} - \xi^k_{i,r}} q^k_{rn+j}, \quad x \in [\xi^k_{i,r}, \xi^k_{j,r}] \tag{2.39}$$

and $h^k(x)$ by

$$h^k(x)|_{[\xi^k_{i,r}, \xi^k_{i+1,r}]} = h^{r,k}_{i,i+1}(x).$$

Then, the piecewise linear curve $\{(x, h^k(x)) : x \in [a, b]\}$ is called the kth level control polygon of f. Now we are in a position to derive a rate of convergence of $\{D_k(r)\}$.

Theorem 2.15. *For $k > \log_2(\frac{nl}{z})$ and $r = 0, 1, \ldots, 2^k - 1$, there exists a constant C such that*

$$D_k(r) \leq \frac{C\,k}{2^k}.$$

Moreover if $\gamma = 0$ then there exist a constant C such that

$$D_k(r) \leq \frac{C}{2^k}.$$

Therefore $\lim_{k\to\infty} D_k(r) = 0$.

Proof: Let $k > \log_2(\frac{nl}{z})$ and $0 \leq r \leq 2^k - 1$. Suppose $0 \leq i < j \leq n$ and $0 \leq u < v \leq n$, and $x \in [\xi^k_{i,r}, \xi^k_{j,r}]$ and $y \in [\xi^k_{u,r}, \xi^k_{v,r}]$. Then we have

$$\begin{aligned} \left\|(x, f^{r,k}_{i,j}(x)) - (y, f^{r,k}_{u,v}(x))\right\|^2 &= |x - y|^2 + |f^{r,k}_{i,j}(x) - f^{r,k}_{u,v}(x)|^2 \\ &\leq \frac{l^2}{2^{2k}} + |f^{r,k}_{i,j}(x) - f^{r,k}_{u,v}(y)|^2. \end{aligned} \tag{2.40}$$

Moreover

$$\begin{aligned} &|f^{r,k}_{i,j}(x) - f^{r,k}_{u,v}(x)| \\ &\leq |f^{r,k}_{i,j}(x) - h^{r,k}_{i,j}(x)| + |h^{r,k}_{i,j}(x) - h^{r,k}_{u,v}(y)| + |h^{r,k}_{u,v}(y) - f^{r,k}_{u,v}(y)|. \end{aligned} \tag{2.41}$$

Since $h^{r,k}_{i,j}(x)$ and $h^{r,k}_{u,v}(y)$ lie in the convex hull of points $\{q^k_{rn+i}\}_{i=0}^n$ the middle term becomes

$$|h^{r,k}_{i,j}(x) - h^{r,k}_{u,v}(y)| \leq n \max_{0\leq i\leq n-1} |q^k_{rn+i+1} - q^k_{rn+i}| \leq nS(k).$$

By Theorem 2.14 we get,

$$|h^{r,k}_{i,j}(x) - h^{r,k}_{u,v}(y)| \leq \begin{cases} nC_s\,\frac{k}{2^k}, & \text{for any } \gamma, \\ nC_s\,\frac{1}{2^k}, & \text{when } \gamma = 0. \end{cases} \tag{2.42}$$

Further

$$
\begin{aligned}
&|f^{r,k}_{i,j}(x) - h^{r,k}_{i,j}(x)| \\
&= |\frac{d(n(x-\xi^k_{j,r}))}{d(n(\xi^k_{i,r}-\xi^k_{j,r}))} q^k_{rn+i} + \frac{d(n(x-\xi^k_{i,r}))}{d(n(\xi^k_{j,r}-\xi^k_{i,r}))} q^k_{rn+j} - \frac{\xi^k_{j,r}-x}{\xi^k_{j,r}-\xi^k_{i,r}} q^k_{rn+i} - \frac{x-\xi^k_{i,r}}{\xi^k_{j,r}-\xi^k_{i,r}} q^k_{rn+j}| \\
&\leq C_q \left\{ |\frac{d(n(x-\xi^k_{j,r}))}{d(n(\xi^k_{i,r}-\xi^k_{j,r}))} - \frac{\xi^k_{j,r}-x}{\xi^k_{j,r}-\xi^k_{i,r}}| + |\frac{d(n(x-\xi^k_{i,r}))}{d(n(\xi^k_{j,r}-\xi^k_{i,r}))} - \frac{x-\xi^k_{i,r}}{\xi^k_{j,r}-\xi^k_{i,r}}| \right\}.
\end{aligned}
$$

By Lemma 2.7 we get,

$$|f^{r,k}_{i,j}(x) - h^{r,k}_{i,j}(x)| \leq C_q C_4 \frac{l}{2^k}. \tag{2.43}$$

By substituting the estimates (2.42) and (2.43) in (2.41) and by (2.40) we get

$$\left\|(x, f^{r,k}_{i,j}(x)) - (y, f^{r,k}_{u,v}(y))\right\| \leq \begin{cases} C\frac{k}{2^k}, & \text{for any } \gamma \\ C\frac{1}{2^k}, & \text{for } \gamma = 0, \end{cases}$$

where $C = l^2 + n\,C_s + 2C_q\,C_4\,l$. This proves the theorem. □

Remark: Thus it is clear from the previous theorem that the sequence $\{f^k(x)\}$ converges to $f(x)$ uniformly on $[a, b]$. Besides, in view of the inequality

$$\left|f^{r,k}_{i,j}(x) - h^{r,k}_{i,j}(x)\right| \leq C_q C_4 \frac{l}{2^k}$$

(see (2.43)) the sequence $\{h^k\}$ also converges to f uniformly.

Note that $f(x)|_{[a^k_r, a^k_{r+1}]} = s_{k,r}(x)$. Therefore $f(x)$ is a $\mathcal{D}$-polynomial spline function as discussed by Gonsor and Neamtu in [38]. Hence by Theorem 7.2 of [38] the following important consequence of quadratic convergence of control curves follows.

Theorem 2.16. *For $k \geq 0$ and $x \in [a, b]$*

$$|f^k(x) - f(x)| \leq K \frac{l^2}{2^{2k}} \tag{2.44}$$

where

$$K = \frac{n^2}{8} \sup_{\substack{\eta_1, \ldots, \eta_n \in [a,\, b] \\ 1 \leq i,j \leq n}} \left\{ \left| \frac{\partial^2 F}{\partial y_i \partial y_j}(\eta_1, \ldots, \eta_n) \right| \right\} \tag{2.45}$$

and F is the polar form of f [38]. Hence the sequence of control curves $\{f^k\}$ converges to f uniformly and quadratically in $[a, b]$.

2.6 Applications and Examples

In this section we present some examples, illustrating how the above scheme can be applied to modify the shape of the curves and surfaces by suitable choices of $d(x)$. This results in great advantage for design purposes.

In case of classical Bezier curve:

$$\sum_{i=0}^{n} P_i\, B_i^n(x; [a,b]), \quad x \in [a,b] \tag{2.46}$$

the tangent vectors of the curve are fixed and are $n(P_1 - P_0)/(b-a)$ and $n(P_n - P_{n-1})/(b-a)$ at the end points P_0 and P_n respectively. However, for a given $d(x)$ the subdivision algorithm generates a GBB curve having tangent vectors:

$$\alpha = n(P_1 \frac{1}{d(b-a)} + P_0 \frac{d'(a-b)}{d(a-b)}), \quad \beta = n(P_n \frac{d'(b-a)}{d(b-a)} + P_1 \frac{1}{d(a-b)})$$

at the end points P_0 and P_1 respectively. Consequently, for different choices of $d(x)$, this algorithm generates GBB curves with different slopes and curvatures near the end points. Therefore, appropriate choice of $d(x)$ can be used for designing curves and tensor product surfaces. In this sense, one can consider the parameters of the equation (2.1) leading to $d(x)$ as design parameters. We illustrate this with some examples.

In the following examples we take a fixed set of control points,

$$P_0 = (1,1),\ P_1 = (0.5,2),\ P_2 = (2.5,2),\ P_3 = (2,1)$$

and $l = 0.5$. Four iterations of the subdivision algorithm are used and the control polygon at fourth level is used to represent the GBB curve. Different GBB curves are displayed in Figure 2.5(a)-2.5(f). Corresponding $d(x)$ and the tangent vectors at the end points of the GBB curves are listed in Table 2.1.

The subdivision scheme for curves is extended to the tensor product scheme by the usual procedure. One advantage of this scheme is that we now have different design variables $d(x)$ in the different directions.

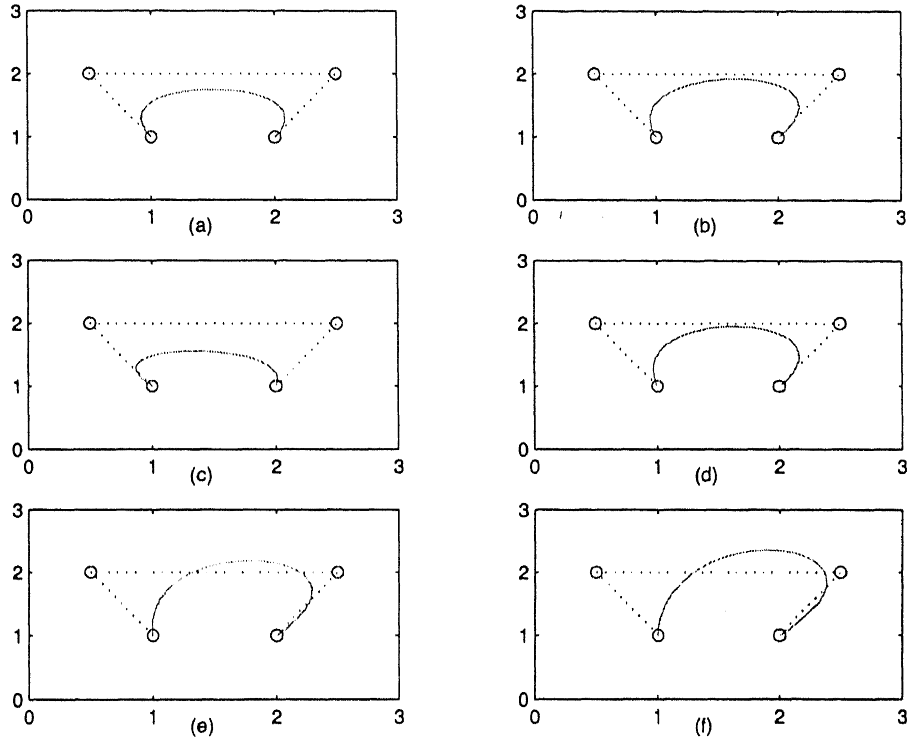


Figure 2.5: GBB curves with different end point tangent vectors

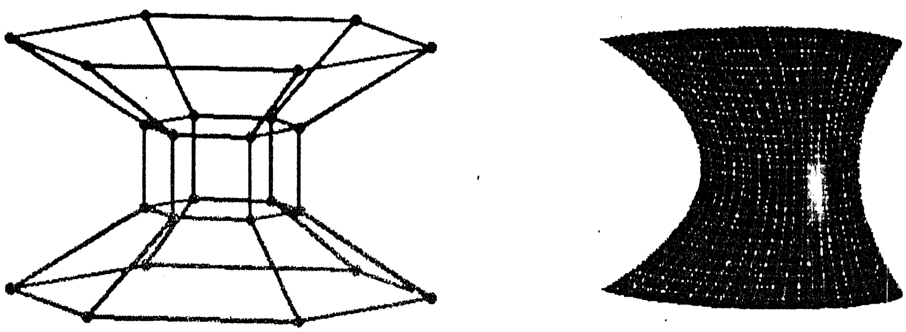
Table 2.1

Figure 2.5	Tangent at P_0	Tangent at P_3	$d(x)$
(a)	(-3, 6)	(-3, -6)	x
(b)	(-171.88, 343.79)	(-171.92, -343.79)	$\sin(x)$
(c)	(-3.61, 5.02)	(-1.41, -5.02)	$\sinh(x)$
(d)	(-1.70, 3.75)	(-39.82, -13.53)	$x\,e^x$
(e)	(-236.13, 76.36)	(-722.18, -785.62)	$e^x \sin(x)$
(f)	(-1.75, 3.49)	(-4.75, -9.5)	$e^x \sinh(x)$

To illustrate the tensor product scheme we take a set of control points suitable for a tensor product scheme. They form vertices of a polyhedron which is shown in Figure 2.6(a). Figure 2.6(b) represents the classical tensor product Bezier surface where as the Figure 2.7-2.8 show different tensor product GBB surfaces obtained by using the above subdivision scheme. We take only three iterations along both X and Y. The values assigned to $d(x)$ along X and Y directions are shown in Table 2.2.

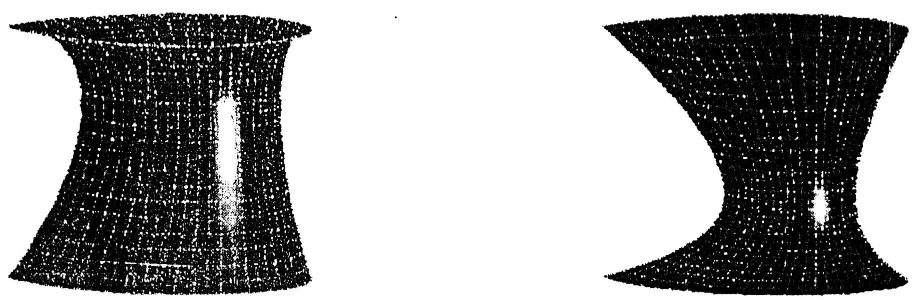
Table 2.2

Figure no.	$d(x)$ along X-axis	$d(x)$ along Y-axis
2.6(b)	x	x
2.7(a)	x	$\sin(x)$
2.7(b)	x	$\sinh(x)$
2.8(a)	$\sin(x)$	x
2.8(b)	xe^x	$\sin(x)$



(a) (b)

Figure 2.6: Original polyhedron and the classical tensor product Bezier surface



(a) (b)

Figure 2.7: Tensor product GBB surfaces for $(x, \sin(x))$ and $(x, \sinh(x))$

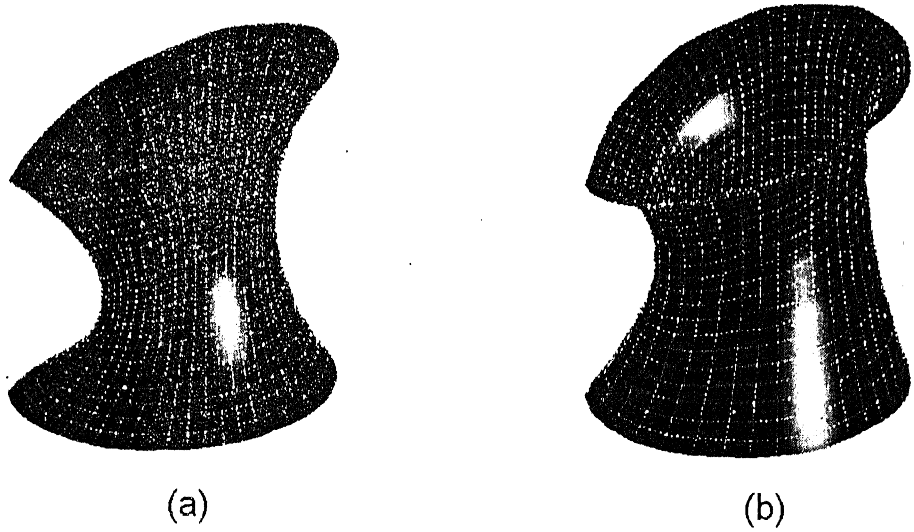


Figure 2.8: Tensor product GBB surfaces for $(\sin(x), x)$ and $(xe^x, \sin(x))$, viewpoint (-78,28)

Chapter 3

Trigonometric Spline Curves

In this chapter a nonstationary subdivision scheme for trigonometric spline curves is introduced. The scheme generalizes the Lane-Riesenfeld scheme for the polynomial spline curves. The chapter is organized into five sections. In Section 3.1 trigonometric splines are introduced briefly. The subdivision scheme is introduced in Section 3.2. The subdivision algorithm is based on a convolution formula which is derived in Section 3.3. The associated refinement equation is proved in Section 3.4. Convergence of the scheme is shown in Section 3.5. Finally in Section 3.6 the scheme is used to reconstruct circles.

3.1 Trigonometric Spline

Trigonometric B-splines $\{T_j^n(x;l) : j = 0, 1, 2, \ldots, m\}$ of order n (with the mesh size l) associated with the knot sequence

$$\Delta := \{t_i = il \ : \ i = 0, 1, \ldots, m+n\}, \qquad m > n, \ \ 0 < l < \frac{\pi}{n} \tag{3.1}$$

are defined in [48] by the following recurrence relation:

$$T_0^1(x;l) = \begin{cases} 1 & \text{if } x \in [0, l), \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

and for $n > 1$,

$$\begin{aligned} T_0^n(x;l) &= \frac{s(x)}{s(t_{n-1})} T_0^{n-1}(x;l) + \frac{s(t_n - x)}{s(t_n - t_1)} T_0^{n-1}(x - l; l) \\ &= \frac{1}{s((n-1)l)} \Big\{ s(x) T_0^{n-1}(x;l) + s(t_n - x) T_0^{n-1}(x - l; l) \Big\} \end{aligned} \tag{3.3}$$

where $s(x) = \sin(x)$. Moreover define

$$T_j^n(x;l) := T_0^n(x - jl; l), \quad \text{for} \quad j = 1, 2, \ldots, m. \tag{3.4}$$

The trigonometric B-spline $T_j^n(x;l)$ is supported on $[t_j,\, t_{j+n}]$ and

$$\{T_j^n(x;l) : j = 0, 1, \ldots, m\}$$

are linearly independent on the interval $[t_{n-1},\, t_{m+1}]$. It is known [48] that on $[t_{n-1}, t_{m+1}]$ any trigonometric spline $f(x)$ of order n with knot sequence Δ has a unique representation:

$$f(x) = \sum_{j=0}^{m} p_j T_j^n(x;l), \qquad p_j \in \mathbb{R},\ j = 1, 2, \ldots, m. \tag{3.5}$$

Control curve and convex hull of trigonometric spline:

For $n > 1$, let

$$\mathbb{L}_n := \operatorname{span}\{\sin((n-1)x),\, \cos((n-1)x)\}. \tag{3.6}$$

For the knot sequence Δ, let the moving knot averages t_i^* are given by

$$t_i^* := \frac{1}{n-1} \sum_{j=i+1}^{i+n-1} t_j = (i + \frac{n}{2})l, \qquad 0 \le i \le m. \tag{3.7}$$

Then, the points $(t_i^*,\, p_i)$, $i = 0, 1, \ldots, m$ are called the *control points* of the trigonometric spline $f(x)$, $x \in [t_{n-1},\, t_{m+1}]$, defined in (3.5).

It is easily computed that for any two successive control points $(t_i^*,\, p_i)$ and $(t_{i+1}^*,\, p_{i+1})$, $i = 0, 1, \ldots, m-1$, the unique function $g_i(t)$ in $\mathbb{L}_n$ interpolating p_i and p_{i+1} at t_i^* and t_{i+1}^* is given as follows,

$$g_i(t) := \frac{s((n-1)(t_{i+1}^* - t))}{s((n-1)(t_{i+1}^* - t_i^*))} p_i + \frac{s((n-1)(t - t_i^*))}{s((n-1)(t_{i+1}^* - t_i^*))} p_{i+1}. \tag{3.8}$$

The function $g(t)$ on $[t_0^*,\, t_m^*]$, where g is defined piecewise by

$$g\big|_{[t_i^*,\, t_{i+1}^*]}(t) = g_i(t),\ i = 0, 1, \ldots, m-1$$

is called the *control curve* of the trigonometric spl ne function $f(x)$.

Definition 3.1. (*Trigonometrically convex set*) [48]):
A subset B of $\mathbb{R}^2$ is called *trigonometrically convex* of order n, $n \geq 2$ if for any two points (ξ_1, c_1) and (ξ_2, c_2) of B with $0 < \xi_2 - \xi_1 < \frac{\pi}{n-1}$, the curve

$$\left\{\left(\xi, \frac{s((n-1)(\xi_2-\xi))}{s((n-1)(\xi_2-\xi_1))}c_1 + \frac{s((n-1)(\xi-\xi_1))}{s((n-1)(\xi_2-\xi_1))}c_2\right) : \xi \in (\xi_1, \xi_2)\right\} \subset B.$$

Definition 3.2. (*Trigonometric convex hull*) [48]:
The *trigonometric convex hull* of order n of a subset B of $\mathbb{R}^2$ is the smallest trigonometrically convex set of order n containing B.

Suppose $S(t)$ is a trigonometric spline function on an interval I with $\{(s_i^*, c_i)\}_{i=0}^{n-1}$ as its associated control points. Then it is shown in [48] that the spline curve $\{(t, S(t)) : t \in I\}$ lies in the trigonometric convex hull of order n of $\{(s_i^*, c_i)\}_{i=0}^{n-1}$. In particular the curve $\{(x, f(x)) : x \in [t_{n-1}, t_{m+1}]\}$, where $f(x) = \sum_{j=0}^{m} p_j T_j^n(x, l)$ lies in the trigonometric convex hull of order n of $\{(t_j^*, p_j) : j = 0, 1, \ldots, m\}$.

3.2 Subdivision Algorithm

In this section we present the subdivision algorithm for the evaluation of the trigonometric spline function $f(x)$ given by (3.5). We denote $\frac{s(j\frac{l}{2})}{s(jl)}$ by $w(j, l)$.

Algorithm 3.1. *Let $n \geq 1$ and $n = 2k-1$ or $n = 2k$ for some $k \geq 1$. Given the control points q_i^0, $i = 0, 1, \ldots, m$ we obtain the control points q_i^1, $i = n-1, \ldots, 2m+1$ as follows: Set*

$$p_{2i}^1 = \begin{cases} q_i^0, & i = 0, 1, \ldots, m \quad \text{if } n = 2k-1 \\ \frac{s(\frac{l}{2})}{s(l)}(q_{i-1}^0 + q_i^0), & i = 1, 2, \ldots, m \quad \text{if } n = 2k, \end{cases} \tag{3.9}$$

$$p_{2i+1}^1 = q_i^0, \quad i = 0, 1, \ldots, m \; (n = 2k \text{ or } n = 2k-1). \tag{3.10}$$

For $j = 2, 3, \ldots k$, define

$$p_i^j = \begin{cases} w(2j-3, l)\{w(2j-2, l)\, p_{i-2}^{j-1} + p_{i-1}^{j-1} + w(2j-2, l)\, p_i^{j-1}\}, \\ \quad \text{for } n = 2k-1, \text{ and } i = 2j-2,\, 2j-1, \ldots, 2m+1. \\ w(2j-2; l)\{w(2j-1; l)\, p_{i-2}^{j-1} + p_{i-1}^{j-1} + w(2j-1; l)\, p_i^{j-1}\}, \\ \quad \text{for } n = 2k, \text{ and } i = 2j-1,\, 2j, \ldots, 2m+1. \end{cases} \tag{3.11}$$

Set $q_i^1 = p_i^k$, $i = n-1, \ldots, 2m+1$.

The equations (3.9) and (3.10) are considered as the doubling up step and (3.11) is considered as the averaging step of the above algorithm. In the next theorem we show that for each i, q_i^1 is a weighted sum of q_j^0, $j = 0, 1, 2, \ldots, m$.

Theorem 3.1. *For each* $i = n-1, \ldots, 2m+1$ *the following refinement relation holds:*

$$q_i^1 = \sum_{j \in \mathbb{Z}} a_{i-2j,n}(l)\, q_j^0 \tag{3.12}$$

where $a_{j,n} \neq 0$ *only for* $j = 0, 1, \ldots, n$. *These coefficients are defined by the recurrence relation*

$$a_{0,1}(l) = 1 = a_{1,1}(l), \quad a_{j,2}(l) = \begin{cases} \frac{s(l/2)}{s(l)}, & j = 0, 2, \\ 1, & j = 1, \end{cases} \tag{3.13}$$

and for $n > 2$,

$$\begin{aligned} a_{j,n}(l) = \frac{s((n-2)\frac{l}{2})}{s((n-2)l)} \frac{s((n-1)\frac{l}{2})}{s((n-1)l)} \Big\{ & a_{j,n-2}(l) + \frac{s((n-1)l)}{s((n-1)\frac{l}{2})} a_{j-1,n-2}(l) \\ & + a_{j-2,n-2}(l) \Big\}. \end{aligned} \tag{3.14}$$

Proof: We prove the theorem by induction on n. In case $n = 1$, by the above algorithm we have $q_{2i}^1 = q_i^0$ and $q_{2i+1}^1 = q_i^0$. Therefore $a_{0,1}(l) = 1 = a_{1,1}(l)$ and $a_{i,1}(l) = 0$ for $i \neq 0, 1$. In case $n = 2$, by the above algorithm we have $q_{2i}^1 = \frac{s(\frac{l}{2})}{s(l)}(q_{i-1}^0 + q_i^0)$ and $q_{2i+1}^1 = q_i^0$. Therefore (3.12) holds true and $a_{i,2}(l) = 0$ for $i \neq 0, 1, 2$.

Let us assume that the theorem holds for $n = m-2$ where $m = 2k$ or $2k-1$. Then for a fixed i, we have

$$p_i^{k-1} = \sum_{j \in \mathbb{Z}} a_{i-2j,m-2}(l)\, q_j^0. \tag{3.15}$$

By (3.11) we have

$$p_i^k = w(m-2,l)\{w(m-1,l)\, p_{i-2}^{k-1} + p_{i-1}^{k-1} + w(m-1,l)\, p_i^{k-1}\}.$$

Substituting (3.15) in the above equation and simplifying we get

$$\begin{aligned} p_i^k = \sum_{j \in \mathbb{Z}} w(m-2,l) \Big\{ & w(m-1,l)\, a_{i-2-2j,m-2}(l) \\ & + a_{i-2j-1,m-2}(l) + w(m-1,l)\, a_{i-2j,m-2}(l) \Big\}\, q_j^0. \end{aligned}$$

Hence (3.12) is satisfied for $n = m$ with

$$a_{i-2j,m} = w(m-2,l)\Big\{w(m-1,l)a_{i-2-2j,m}(l) + a_{i-2j-1,m}(l) + w(m-1,l)a_{i-2j,m}(l)\Big\}q_j^0.$$

This proves (3.14) for $n = m$. It is clear that $a_{i,m} = 0$, $i = 0, 1, \ldots, m$. This completes the proof of the theorem. □

Repeated use of the algorithm outlined above for the generation of new control points, gives us at the $r+1$th level the control points

$$\{q_i^{r+1} \,:\, i = (2^{r+1}-1)(n-1), \ldots, 2^{r+1}(m+1)-1\}$$

from $\{q_i^r \,:\, i = (2^r-1)(n-1), \ldots, 2^r(m+1)-1\}$ by the rule:

$$q_i^{r+1} = \sum_{j\in\mathbb{Z}} a_{i-2j,n}(\frac{l}{2^r})\, q_j^r. \tag{3.16}$$

Clearly, this subdivision algorithm is uniform non-stationary and binary [35]. The set of coefficients

$$\mathrm{a}^{(\mathrm{r})} = \{a_{j,n}(\frac{l}{2^r}) \,:\, j = 0, 1, \ldots, n\}$$

is the mask of the subdivision scheme at the r-th level.

So far we have presented an algorithm for the generation of new control points using a mask $\mathbf{a}^{(\mathbf{r})}$. It will be shown later in Section 3.5 that the following associated refinement equation holds,

$$\sum_{i=0}^{m} q_i^0\, T_i^n(x;l) = \sum_{i=n-1}^{2m+1} q_i^1\, T_i^n(x;\frac{l}{2}), \quad x \in [t_{n-1}, t_{m+1}] \tag{3.17}$$

where $\{T_i^n(x;\frac{l}{2}) \,:\, i = n-1 \ldots 2m+1\}$ are trigonometric B-splines (with the mesh size $l/2$) associated with the knot sequence

$$\Delta_1 := \{t_{i,1} = i\frac{l}{2} \,:\, i = n-1, n, \ldots, 2m+n+1\}. \tag{3.18}$$

At the r-th level of refinement, we get

$$\sum_{i=0}^{m} q_i^0\, T_i^n(x;l) = \sum_{i=(2^r-1)(n-1)}^{2^r(m+1)-1} q_i^r\, T_i^n(x;\frac{l}{2^r}), \quad x \in [t_{n-1}, t_{m+1}] \tag{3.19}$$

where $\{T_i^n(x; \frac{l}{2^r}) : i = (2^r-1)(n-1)\dots 2^r(m+1)-1\}$ are trigonometric B-splines (with mesh size $\frac{l}{2^r}$) associated with the knot sequence:

$$\Delta_r := \{t_{i,r} = i\frac{l}{2^r} : i = (2^r-1)(n-1), \dots, 2^r(m+1)-1+n\}. \tag{3.20}$$

Thus we have obtained a subdivision algorithm, a corresponding set of masks and refinement equations. Hence we have a uniform non-stationary binary subdivision scheme(BSS).

We note here that generalization of Oslo algorithm to trigonometric splines with uniform knots [48] satisfy the same refinement equations (3.17) and (3.19), and has the same doubling up step as in Algorithm 3.1. However, the present algorithm is distinct from the Oslo algorithm in view of its averaging process.

3.3 Convolution Formula for Trigonometric B-spline

In this section we derive a convolution formula for trigonometric B-splines which will be used to prove the refinement equation in the next section. An analogous formula for classical B-spline can be found in [20, 21, 42, 71]. The Fourier transform $f^\frown$ of the function $f \in L^1(\mathbb{R})$ is defined by

$$f^\frown(t) = \int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dx, \quad t \in R.$$

For any $m, n \in N$ such that $n-m$ is even we denote $x(m)\,x(m+2)\dots x(n)$ by $\bar{\prod}_{j=m}^{n} x(j)$.

The following lemma gives an explicit expression for $(T_0^n(\cdot; l))^\frown(y)$.

Lemma 3.2. *For $n \geq 1$ we have,*

$$(T_0^n(\cdot; l))^\frown(y) = 2^n(n-1)!\; K(n)\; e^{-i(ny)\frac{l}{2}} \bar{\prod}_{j=-n+1}^{n-1} \frac{s((y-j)\frac{l}{2})}{y-j}, \quad y \in \mathbb{R}, \tag{3.21}$$

where

$$K(n) = \begin{cases} 1, & n = 1 \\ \frac{K(n-1)}{s((n-1)l)}, & n > 1. \end{cases} \tag{3.22}$$

Proof. We prove the lemma by induction on n. For the sake of convenience we employ the notations T_0^i and $T_0^i(\cdot - l)$ for $T_0^i(\cdot; l)$ and $T_0^i(\cdot - l; l)$ respectively. It is easy to derive that

$$(T_0^1)^\frown(y) = \frac{2}{y}\, e^{-i\frac{l}{2}y}\, s(y\frac{l}{2}).$$

This shows that (3.21) is true for $n = 1$.

For $n \geq 1$, from recurrence relation (3.3) we have

$$(T_0^{n+1})^\frown(y) = \frac{1}{s(nl)}\, \{(s\, T_0^n)^\frown(y) + (s(t_{n+1} - \cdot)\, T_0^n(\cdot - l))^\frown(y)\}. \tag{3.23}$$

Since

$$(s\, g)^\frown(y) = \frac{g^\frown(y-1) - g^\frown(y+1)}{2i}, \quad (s(a - \cdot)\, g)^\frown(y) = \frac{e^{ia}\, g^\frown(y+1) - e^{-ia}\, g^\frown(y-1)}{2i}$$

and $(g(\cdot - l))^\frown(y) = e^{-il} g^\frown(y)$ for any $g \in L^1$, we have

$$\begin{aligned}
&(T_0^{n+1})^\frown(y)\\
&= \frac{1}{s(nl)}\left\{\frac{(T_0^n)^\frown(y-1) - (T_0^n)^\frown(y+1)}{2i} + \frac{e^{-i(y-n)l}\,(T_0^n)^\frown(y+1)}{2i} - \frac{e^{-i(y+n)l}\,(T_0^n)^\frown(y-1)}{2i}\right\}\\
&= \frac{1}{s(nl)}\left\{\frac{1 - e^{-i(y+n)l}}{2i}\,(T_0^n)^\frown(y-1) + \frac{e^{-i(y-n)l} - 1}{2i}\,(T_0^n)^\frown(y+1)\right\},\\
&= \frac{1}{s(nl)}\left\{e^{-i(y+n)\frac{l}{2}}\, s((y+n)\frac{l}{2})\,(T_0^n)^\frown(y-1) - e^{-i(y-n)\frac{l}{2}}\, s((y-n)\frac{l}{2})\,(T_0^n)^\frown(y+1)\right\}.
\end{aligned} \tag{3.24}$$

By taking $n = 1$ we get

$$(T_0^2)^\frown(y) = 2^2\, \frac{e^{-iyl}}{s(l)} \prod_{j=-1}^{1} \frac{s((y-j)\frac{l}{2})}{s(y-j)}$$

which implies that (3.21) also holds for $n = 2$.

Let us assume that (3.21) holds for $n = m$. Then by (3.24) we have

$$\begin{aligned}
&(T_0^{m+1})^\frown(y)\\
&= \frac{1}{s(ml)}\, 2^m (m-1)! \Big\{ e^{-i(y+m)\frac{l}{2}}\, e^{-im(y-1)\frac{l}{2}}\, s((y+m)\frac{l}{2})\, K(m) \prod_{-m+1}^{m-1} \frac{s((y-j-1)\frac{l}{2})}{y-j-1}\\
&\quad - e^{-i(y-m)\frac{l}{2}}\, e^{-im(y+1)\frac{l}{2}}\, s((y-m)\frac{l}{2})\, K(m) \prod_{-m+1}^{m-1} \frac{s((y-j+1)\frac{l}{2})}{y-j+1} \Big\}
\end{aligned}$$

$$= \frac{K(m)}{s(ml)} 2^m (m-1)!\, e^{-i(m+1)y\frac{l}{2}} \prod_{-m}^{m} \frac{s((y-j)\frac{l}{2})}{y-j} (y+m-y+m)$$

$$= K(m+1)\, 2^{m+1}\, m!\, e^{-i(m+1)y\frac{l}{2}} \prod_{-m}^{m} \frac{s((y-j)\frac{l}{2})}{y-j}.$$

Which proves (3.21) for $n = m+1$. Hence the lemma follows by induction. □

Now we present an expression for the trigonometric B-splines as a convolution product, analogous to corresponding result for the polynomial B-splines.

Proposition 3.3. *For $n > 2$, we have*

$$T_0^n(x;l) = \Lambda_n(x;l) * T_0^{n-2}(x;l), \qquad x \in \mathbb{R} \tag{3.25}$$

where,

$$\Lambda_n(x;l) = \begin{cases} \frac{n-2}{s((n-2)l} \frac{s((n-1)x)}{s((n-1)l)}, & 0 \le x \le l \\ \frac{n-2}{s((n-2)l)} \frac{s(n-1)(2l-x))}{s((n-1)l)}, & l \le x \le 2l \\ 0, & \textit{elsewhere}. \end{cases} \tag{3.26}$$

Moreover

$$T_i^n(x;l) = \Lambda_n(x;l) * T_i^{n-2}(x;l),\ i \in \mathbb{Z},\ x \in \mathbb{R}. \tag{3.27}$$

Proof. From the previous lemma we have

$$\frac{(T_0^n)^\frown(y)}{(T_0^{n-2})^\frown(y)} = \frac{2^2\,(n-1)(n-2)\,e^{-iyl}}{s((n-2)l)\,s((n-1)l)} \frac{s((y-n+1)\frac{l}{2})s((y+n-1)\frac{l}{2})}{(y-n+1)(y+n-1)}. \tag{3.28}$$

Let us denote

$$h_1(x) = \chi_{[0,l]}(x)\, e^{i(n-1)x} \quad \text{and} \quad h_2(x) = \chi_{[0,l]}(x)\, e^{-i(n-1)x}$$

where $\chi_{[0,l]}$ is the characteristic function of $I = [0,l]$. Clearly

$$(h_1)^\frown(y) = 2\,e^{-i(y-n+1)\frac{l}{2}} \frac{s((y-n+1)\frac{l}{2})}{y-n+1},$$

and

$$(h_2)^\frown(y) = 2\,e^{-i(y+n-1)\frac{l}{2}} \frac{s((y+n-1)\frac{l}{2})}{y+n-1}.$$

Hence

$$(h_1 * h_2)^\frown(y) = 2^2\, e^{-iyl}\, \frac{s((y-n+1)\frac{l}{2})}{y+n-1}\, \frac{s((y+n-1)\frac{l}{2})}{y+n-1}.$$

Thus from (3.28) we get

$$(T_0^n)^\frown(y) = \frac{n-1}{s((n-1)l)}\, \frac{n-2}{s((n-2)l)}\, (h_1 * h_2)^\frown(y)\, (T_0^{n-2})^\frown(y).$$

Since $T_0^n(.;l)$ and $h_1 * h_2$ are continuous functions we have

$$T_0^n(x;l) = \frac{(n-1)(n-2)}{s((n-1)l)(s(n-2)\,l)}(h_1 * h_2 * T_0^{n-2})(x;l) \quad x \in \mathbb{R}. \tag{3.29}$$

Let us now calculate $(h_1 * h_2)(x)$:

$$\begin{aligned}
(h_1 * h_2)(x) &= \int_{-\infty}^{\infty} h_1(y)\, h_2(x-y)\,dy \\
&= \int_0^l e^{i(n-1)y} \chi_{[0,l]}(x-y)\, e^{-i(n-1)(x-y)}\, dy \\
&= e^{-i(n-1)x} \int_{[0,l]\,\bigcap[x-l,x]} e^{i2(n-1)y}\, dy.
\end{aligned}$$

If $0 \le x \le l$ then $[0,l] \bigcap [x-l,x] = [0,x]$. Therefore when $0 \le x \le l$, we have

$$\begin{aligned}
(h_1 * h_2)(x) &= e^{-i(n-1)x} \int_0^x e^{i2(n-1)y}\, dy \\
&= e^{-i(n-1)x} \left[\frac{e^{i2(n-1)y}}{i2(n-1)}\right]_0^x \\
&= e^{-i(n-1)x} \left\{\frac{e^{i2(n-1)x} - 1}{i2(n-1)}\right\} \\
&= \frac{s((n-1)x)}{n-1}.
\end{aligned}$$

If $l \le x \le 2l$ then $[0,l] \bigcap [x-l,x] = [x-l,l]$. Therefore, when $l \le x \le 2l$, we have

$$\begin{aligned}
(h_1 * h_2)(x) &= e^{-i(n-1)x} \int_{x-l}^{l} e^{i2(n-1)y}\, dy \\
&= e^{-i(n-1)x} \left[\frac{e^{i2(n-1)y}}{i\,2(n-1)}\right]_{x-l}^{l} \\
&= e^{-i(n-1)x} \left\{\frac{e^{i2(n-1)l} - e^{i\,2(n-1)(x-l)}}{i\,2(n-1)}\right\} \\
&= \frac{s((n-1)(2l-x))}{n-1}.
\end{aligned}$$

If $x \in [0,2l]^c$ then $[0,l] \bigcap [x-l,l] = \phi$. Therefore, when $x \in [0,2l]^c$, we have

$$(h_1 * h_2)(x) = 0.$$

This proves (3.25). The proof of $T_i^n(x,l) = \Lambda_n(x,l) * T_i^{n-2}(x,l)$ follows by a simple translation of the variable. □

3.4 Refinement Equation

In this section we prove the refinement equation (3.17) associated with the subdivision scheme. We require the following lemma in which $\Lambda_n(x;l)$ is expressed as a linear combination of its binary refinements $\Lambda_n(x;\frac{l}{2})$, $\Lambda_n(x-\frac{l}{2};\frac{l}{2})$, $\Lambda_n(x-l;\frac{l}{2})$.

Lemma 3.4. *For $n > 2$ and $x \in [0,2l]$, we have*

$$\begin{aligned}\Lambda_n(x;l) = &\frac{s((n-2)\frac{l}{2})}{s((n-2)l)}\Big\{\frac{s((n-1)\frac{l}{2})}{s((n-1)l)}\Lambda_n(x;\frac{l}{2}) + \Lambda_n(x-\frac{l}{2};\frac{l}{2}) \\ &+ \frac{s((n-1)\frac{l}{2})}{s((n-1)l)}\Lambda_n(x-l;\frac{l}{2})\Big\}.\end{aligned} \tag{3.30}$$

Proof: We observe that $\Lambda_n(x;l)$, $\Lambda_n(x;\frac{l}{2}) \in \mathbb{L}_n$ so also their translations $\Lambda_n(x-il;l)$ and $\Lambda_n(x-i\frac{l}{2};\frac{l}{2})$. We also observe that $\Lambda_n(x;\frac{l}{2})$ and $\Lambda_n(x-\frac{l}{2};\frac{l}{2})$ are linearly independent on $[0,\ l]$ while $\Lambda_n(x-\frac{l}{2};\frac{l}{2})$ and $\Lambda_n(x-l;\frac{l}{2})$ are linearly independent on $[l,\ 2l]$. Therefore,

$$\Lambda_n(x;l) = a_0\Lambda_n(x;\frac{l}{2}) + a_1\Lambda_n(x-\frac{l}{2};\frac{l}{2}), \quad x \in [0,\ l] \tag{3.31}$$

$$\Lambda_n(x;l) = b_0\Lambda_n(x-\frac{l}{2};\frac{l}{2}) + b_1\Lambda_n(x-l;\frac{l}{2}), \quad x \in [l,\ 2l] \tag{3.32}$$

for some a_0, a_1, b_0 and b_1 in $\mathbb{R}$.

By taking $x = \frac{l}{2}$ in (3.31) we get

$$\frac{n-2}{s((n-2)l)}\frac{s((n-1)\frac{l}{2})}{s((n-1)l)} = a_0\frac{n-2}{s((n-2)\frac{l}{2})}\frac{s((n-1)\frac{l}{2})}{s((n-1)\frac{l}{2})}.$$

Therefore $a_0 = \frac{s((n-2)\frac{l}{2})}{s((n-2)l)}\frac{s((n-1)\frac{l}{2})}{s((n-1)l)}$.

By taking $x = l$ in (3.31) and (3.32) we get

$$\frac{n-2}{s((n-2)l)} = a_1 \frac{n-2}{s((n-2)\frac{l}{2})} \frac{s((n-1)\frac{l}{2})}{s((n-1)\frac{l}{2})}$$
$$\frac{n-2}{s((n-2)l)} = b_0 \frac{n-2}{s((n-2)\frac{l}{2})} \frac{s((n-1)\frac{l}{2})}{s((n-1)\frac{l}{2})}.$$

Therefore $a_1 = \frac{s((n-2)\frac{l}{2})}{s((n-2)l)} = b_0$.

Similarly by taking $x = 3\frac{l}{2}$ in (3.32) we get $b_1 = \frac{s((n-2)\frac{l}{2})}{s((n-2)l)} \frac{s((n-1)\frac{l}{2})}{s((n-1)l)} = a_0$. This proves the lemma. □

We are now in a position to prove the main result of the section, the validity of the refinement equation stated in the following theorem.

Theorem 3.5. *Let $n \geq 1$ and $n = 2k$ or $2k-1$ for some $k \geq 1$. For $x \in [t_{n-1}, t_{m+1}]$ we have*

$$\sum_{i=0}^{m} q_i^0 \, T_i^n(x; l) = \sum_{i=n-1}^{2m+1} p_i^k \, T_i^n(x; \frac{l}{2}) = \sum_{i=n-1}^{2m+1} q_i^1 \, T_i^n(x; \frac{l}{2}) \tag{3.33}$$

where p_i^k is defined by (3.11).

Proof. We prove the theorem by induction with respect to k. Since $T_i^1(x; l) = T_{2i}^1(x; \frac{l}{2}) + T_{2i+1}^1(x; \frac{l}{2})$ for $x \in [t_0, t_{m+1}]$, we have

$$\sum_{i=0}^{m} q_i^0 \, T_i^1(x; l) = \sum_{i=0}^{m} q_i^0 \left(T_{2i}^1(x; \frac{l}{2}) + T_{2i+1}^1(x; \frac{l}{2}) \right) = \sum_{i=0}^{2m+1} p_i^1 \, T_i^1(x; \frac{l}{2}). \tag{3.34}$$

Which proves (3.33) for the case $n = 1$, corresponding to $k = 1$ for $n = 2k - 1$.

For $k = 1$ for the expression $n = 2k$, we have $n = 2$. Then our interval of consideration becomes $[t_1, t_{m+1}] = [l, (m+1)l]$. Since $\sum_{i=0}^{m} q_i^0 \; T_i^2(x; l)$ is also a trigonometric spline of order 2 on $[t_1, t_{m+1}]$ with knot sequence Δ_1, there exists $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, 2m+1$ such that

$$\sum_{i=0}^{m} q_i^0 \; T_i^2(x; l) = \sum_{i=1}^{2m+1} \alpha_i \; T_i^2(x; \frac{l}{2}), \quad x \in [t_1, t_{m+1}].$$

Defi ng $x = (2i+1)\frac{l}{2}$ and $x = (i+1)l = (2i+2)\frac{l}{2}$ in the above equation we get

A si $\frac{(l/2)}{s(l)}\left(q_{i-1}^0 + q_i^0\right)$ and $\alpha_{2i+1} = q_i^0$ respectively. This proves (3.33) for the case

poir

us assume that (3.33) holds for $n = r$ where $r = 2k-1$ or $2k$. Then we have

De

$$\sum_{i=0}^{m} q_i^0\, T_i^r(x;l) = \sum_{i=r-1}^{2m+1} p_i^k\, T_i^r(x;\frac{l}{2}). \tag{3.35}$$

Th

mo prove that (3.33) holds for $n = r+2$ with $r \geq 1$. By (3.27) we have

$$\sum_{0}^{} q_i^0\, T_i^{r+2}(x;l) = \sum_{i=0}^{m} q_i^0\,(\Lambda_{r+2}(x;l) * T_i^r(x;l)) = \Lambda_{r+2}(x;l) * \sum_{i=0}^{m} q_i^0\, T_i^r(x;l)$$

it y induction hypothesis (3.35) gives

t

$$\sum_{i=0}^{m} q_i^0\, T_i^{r+2}(x;l) = \Lambda_{r+2}(x;l) * \sum_{i=r-1}^{2m+1} p_i^k\, T_i^r(x;\frac{l}{2}).$$

t

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ne expression (3.30) for $\Lambda_{r+2}(x;l)$, and taking $s_1 := \frac{s(r\frac{l}{2})}{s(rl)}\frac{s((r+1)\frac{l}{2})}{s((r+1)l)}$ and $s_2 := \frac{s(r\frac{l}{2})}{s(rl)}$

$$q_i^0\, T_i^{r+2}(x;l)$$

$$\left\{s_1\,\Lambda_{r+2}(x;\frac{l}{2}) + s_2\,\Lambda_{r+2}(x-\frac{l}{2};\frac{l}{2}) + s_1\,\Lambda_{r+2}(x-l;\frac{l}{2})\right\} * \sum_{i=r-1}^{2m+1} p_i^k\, T_i^r(x;\frac{l}{2})$$

$$\sum_{=r-1}^{2m+1} p_i^k\left\{s_1\, T_i^{r+2}(x;\frac{l}{2}) + s_2\, T_i^{r+2}(x-\frac{l}{2};\frac{l}{2}) s_1\, T_i^{r+2}(x-l;\frac{l}{2})\right\}.$$

r interval of consideration is $[t_{r+1}, t_{m+1}] = [(r+1)l, (m+1)l]$ and $T_i^{r+2}(x;\frac{l}{2}) = 0$ $-1)l, (m+1)l]$ for $i \leq r$ and $i \geq 2m+2$, we have

$$\begin{aligned}
&\sum_{i=0}^{m} q_i^0\, T_i^{r+2}(x;l)\\
&= \sum_{i=r+1}^{2m+1} s_1\, p_i^k\, T_i^{r+2}(x;\frac{l}{2}) + \sum_{i=r}^{2m} s_2\, p_i^k\, T_{i+1}^{r+2}(x;\frac{l}{2}) + \sum_{i=r-1}^{2m-1} s_1\, p_i^k\, T_{i+2}^{r+2}(x;\frac{l}{2})\\
&= \sum_{i=r+1}^{2m+1}\left\{s_1\, p_i^k + s_2\, p_{i-1}^k + s_1\, p_{i-2}^k\right\} T_i^{r+2}(x;\frac{l}{2})\\
&= \sum_{i=r+1}^{2m+1} p_i^{k+1}\, T_i^{r+2}(x;\frac{l}{2}).
\end{aligned}$$

oves (3.33) for $n = r+2$. Hence the theorem follows by induction. □

3.5 Convergence Analysis

In this section we discuss the convergence aspect of the subdivision scheme. The convergence of the subdivision scheme is proved using the convergence characterization of the subdivision scheme via the convergence of control curves (Theorem 1.1). An alternative proof is also given using the characteristic polynomials of the subdivision scheme, following a procedure in [35].

We follow Definition 1.1 as the definition of convergence of a subdivision scheme. First we show the convergence of the subdivision scheme using the convergence of control curves. A control curve has been defined in Section 3.1 by (3.8). Let f^r denote the control curve at the rth level of the refinement. For its construction we refer to (3.19).

The spline function $f(x)$ has the expression

$$f(x) = \sum_{i=(2^r-1)(n-1)}^{2^r(m+1)-1} q_i^r\, T_i(x; \frac{l}{2^r}), \quad x \in [t_{n-1}, t_{m+1}] \tag{3.36}$$

after the rth refinement. The points

$$(t^*_{i,r}, q_i^r),\ i = (2^r-1)(n-1), \ldots, 2^r(m+1)-1$$

with $t^*_{i,r} = \frac{t_{i+1,r}+\ldots+t_{i+n-1,r}}{n-1}$ are called control points and the function

$$f^r(x) = \sum_{i=(2^r-1)(n-1)}^{2^r(m+1)-1} q_i^r\, \psi_i(x; \frac{l}{2^r}),\ x \in [t^*_{(2^r-1)(n-1),r},\, t^*_{2^r(m+1)-1,r}] \tag{3.37}$$

with

$$\psi_i(x; \frac{l}{2^r}) = \begin{cases} \frac{s(x-t^*_{i-1,r})}{s(\frac{l}{2^r})}, & x \in [t^*_{i-1,r}, t^*_{i,r}), \\ \frac{s(t^*_{i+1,r}-x)}{s(\frac{l}{2^r})}, & x \in [t^*_{i,r}, t^*_{i+1,r}], \\ 0, & else \end{cases}$$

is the corresponding control curve.

Clearly, the functions $\psi_i(x; \frac{l}{2^r}), i \in \mathbb{Z}$ are compactly supported, continous and $\|\psi_i(\cdot; \frac{l}{2^r})\|_\infty \leq 1$ for all $i \in \mathbb{Z}$. Moreover, for $x \in [t^*_{i,r},\, t^*_{i+1,r}]$, it is easily seen that

$$\sum_{i\in\mathbb{Z}} \psi_i(x; \frac{l}{2^r}) = \frac{\cos(\frac{t^*_{i+1,r}-x}{2} + \frac{t^*_{i,r}-x}{2})}{\cos(\frac{l}{2^{r+1}})} \tag{3.38}$$

and $\cos(\frac{t^*_{i+1,r}-x}{2}+\frac{t^*_{i,r}-x}{2}) \to 1$ uniformly as $r \to \infty$. Therefore,

$$\lim_{r\to\infty}\sum_{i\in\mathbb{Z}}\psi_i(\cdot;\frac{l}{2^r}) = 1 \quad \text{uniformly in} \quad \mathbb{R}. \tag{3.39}$$

Since $\psi_i(x;\frac{l}{2^r})$, $i \in \mathbb{Z}$ are compactly supported, by the Remark in Page 51 of [29] the functions $\{\psi_i(x;\frac{l}{2^r}),\ i \in \mathbb{Z}\}$ satisfy the stability condition

$$K_1\|h\|_\infty \le \|h_i\psi_i(\cdot;\frac{l}{2^r})\|_\infty \tag{3.40}$$

where $h := \{h_i\}_{i=0}^{\infty} \in l^\infty(Z)$, (see (4.2) of [35]).

The following theorem is a consequence of Theorem 4.4 of [48], which shows that the sequence of control curves $\{f^r\}$ converges uniformly and quadratically to f in $[t_{n-1},\ t_{m+1}]$.

Theorem 3.6. *Let f and f^r be defined by* (3.36) *and* (3.37) *respectively, then*

$$\|f - f^r\|_{[t_{n-1},t_{m+1}]} \le \frac{K\,l^2}{2^{2r}\cos(\frac{l}{2^{r+1}})}\|Lf\|_{[t_{n-1},t_{m+1}]} \tag{3.41}$$

where $L := D^2 + (n-1)^2$ and K is a constant depends upon n and f.

Now we have all the necessary conditions satisfied for the application of Theorem 1.1 of [35]. Therefore our subdivision scheme converges as a consequence of the theorem.

We now discuss an alternative method for the proof of convergence of the subdivision scheme using the characteristic polynomials of the scheme. Every non-stationary uniform BSS is associated with a set of mask and a set of characteristic polynomials [35].

The characteristic polynomial at the rth level of our subdivision scheme corresponding to the mask $\{a_{i,n}(\frac{l}{2^r}),\ i = 0, 1, \ldots, n\}$ is given by

$$a_n^{(r)}(z) = \sum_{j=0}^{n} a_{j,n}(\frac{l}{2^r})z^j, \quad z = e^{ix},\ i = \sqrt{-1},\ x \in \mathbb{R}. \tag{3.42}$$

Clearly,

$$a_1^{(r)}(z) = 1 + z \tag{3.43}$$

and

$$a_2^{(r)}(z) = \frac{1}{2c(\frac{l}{2^r})}\left(1 + 2c(\frac{l}{2^r})\,z + z^2\right) = \frac{1}{2c(\frac{l}{2^r})}(1 + e^{-i\frac{l}{2^{r+1}}}z)(1 + e^{i\frac{l}{2^{r+1}}}z). \tag{3.44}$$

The characteristic polynomials $a_n^{(r)}(z)$, $n > 2$, are given by the following lemma.

Lemma 3.7. *For $n > 2$*

$$a_n^{(r)}(z) = \begin{cases} \frac{2^{-n+1}}{\prod_{j=1}^{n-1} c(j\frac{l}{2^r+1})} \prod_{j=1}^{m}(1+e^{-ij\frac{l}{2^r}}\, z) \prod_{j=1}^{m}(1+e^{ij\frac{l}{2^r}}\, z)(1+z), & n = 2m+1 \\ \frac{2^{-n+1}}{\prod_{j=1}^{n-1} c(j\frac{l}{2^r+1})} \prod_{j=1}^{m}(1+e^{-i(2j-1)\frac{l}{2^r+1}}\, z) \prod_{j=1}^{m}(1+e^{i(2j-1)\frac{l}{2^r+1}})\, z), & n = 2m \cdot \end{cases} \tag{3.45}$$

where $c(x) = cos(x)$.

Proof: By (3.14)

$$\begin{aligned} a_n^{(r)}(z) &= \frac{2^{-2}}{c((n-2)\frac{l}{2^r+1}\, c((n-1)\frac{l}{2^r+1})}(z^2 + 2c((n-1)\frac{l}{2^r+1})\, z + 1)\, a_{n-2}^{(r)}(z) \\ &= \frac{2^{-2}}{c((n-2)\frac{l}{2^r+1}\, c((n-1)\frac{l}{2^r+1})}(1+e^{-i(n-1)\frac{l}{2^r+1}}\, z)(1+e^{i(n-1)\frac{l}{2^r+1}}\, z)\, a_{n-2}^{(r)}(z). \end{aligned}$$

By repeated use of the above recurrence relation with respect to n, we get

$$a_n^{(r)}(z) = \begin{cases} \frac{2^{-n+1}}{\prod_{j=1}^{n-1} c(j\frac{l}{2^r+1})} \prod_{j=1}^{m}(1+e^{-ij\frac{l}{2^r}}\, z) \prod_{j=1}^{m}(1+e^{ij\frac{l}{2^r}}\, z)\, a_1^{(r)}(z) & n = 2m+1 \\ \frac{2^{-n+1}}{\prod_{j=1}^{n-1} c(j\frac{l}{2^r+1})} \prod_{j=1}^{m}(1+e^{-i(2j-1)\frac{l}{2^r+1}}\, z) \prod_{j=1}^{m}(1+e^{i(2j-1)\frac{l}{2^r+1}}\, z), & n = 2m. \square \end{cases}$$

By Example 1.1 if the characteristic polynomials at r-th level of a non-stationary BSS is

$$b_d^{(r)}(z) = 2\prod_{j=1}^{d} \frac{1}{2}(1 + e^{\frac{c_j}{2^r+1}}\, z),\ r \in \mathbb{Z}_+,\ c_j \in \mathbb{C} \tag{3.46}$$

then the BSS converges and the limit function is in $C^{d-2}(\mathbb{R})$. Further the limit function is an exponential spline.

We use the above in order to show the convergence of our scheme. For this we define

$$c_j = \begin{cases} -i(2j-1)l, & j = 1, \dots, m \\ i(2j-1)l, & j = m+1, \dots, 2m \end{cases}, \quad \text{for} \quad n = 2m,$$

$$c_j = \begin{cases} -2ijl, & j = 1, \dots, m \\ 2ijl, & j = m+1, \dots, 2m, \\ 0, & j = 2m \end{cases}, \quad \text{for} \quad n = 2m+1.$$

Then the characteristic polynomials $\{a_n^r(z)\}$ in our scheme has the form

$$a_n^{(r)}(z) = \frac{1}{\prod_{j=1}^{n-1} c(j\frac{l}{2^r+1})}\, b_n^{(r)}(z). \tag{3.47}$$

Denote $\frac{1}{\prod_{j=1}^{n-1} c(j\frac{l}{2^r+1})}$ by β_r. Then for all k we have

$$\prod_{r=0}^{k} \beta_r = \prod_{j=1}^{n-1} 2^{k+1} \frac{s(j\frac{l}{2^{k+1}})}{s(jl)} \leq (n-1)! \prod_{j=1}^{n-1} \frac{l}{s(jl)}.$$

Since $2^{k+2}s(j\frac{l}{2^{k+2}}) > 2^{k+1}s(j\frac{l}{2^{k+1}})$ for $j = 1, \ldots, n-1$, $0 < l < \frac{\pi}{n}$, $k \in \mathbb{Z}_+$ the sequence $\{\prod_{r=0}^{k} \beta_r\}$ is a monotonically increasing sequence which is bounded above and hence converges. In fact

$$\lim_{k\to\infty} (\prod_{r=0}^{k} \beta_r) = (n-1)! \prod_{j=1}^{n-1} \frac{l}{s(jl)}.$$

Therefore our subdivision scheme converges and the limit function is in $C^{n-2}(\mathbb{R})$.

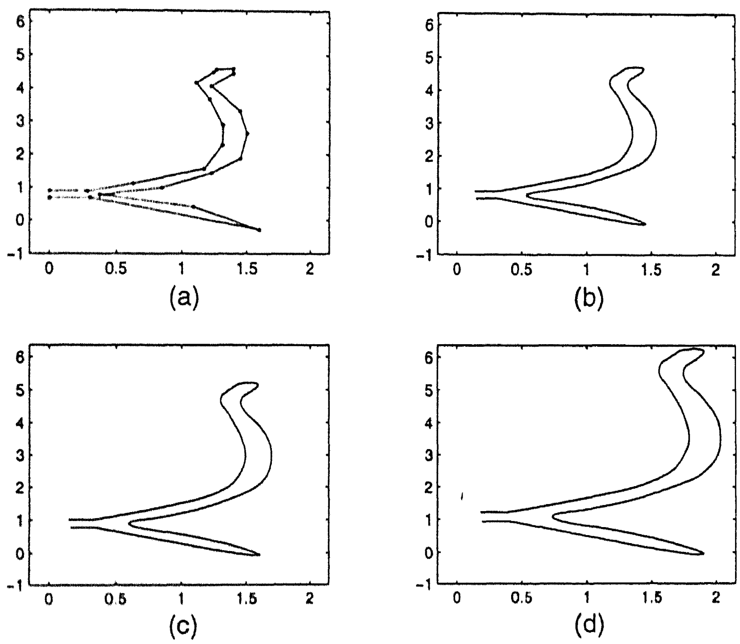


Figure 3.1: Quadratic trigonometric spline: (a) initial control points. Limiting curves after three iterations: (b) l=0.25, (c) l=0.5, (d) l=0.75

Towards this end we illustrate the subdivision algorithm in Figures 3.1 and 3.2. Note that we now have variable l which is now used as a design variable. Taking different value of l we get limiting curves with different curvatures for the same set of control points and control polygon.

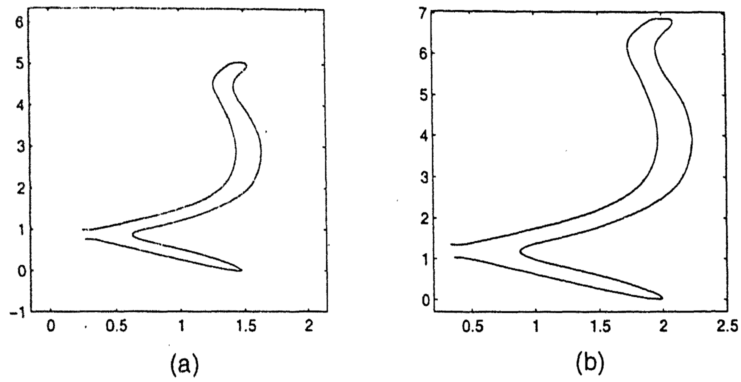


Figure 3.2: Cubic trigonometric spline: Limiting curves after three iterations (a) l=0.25, (b) l=0.5

3.6 Generating Circles

In CAGD we sometimes require to construct circular parts of geometric models. In this section we implement our subdivision scheme to reconstruct circles. For this we take n=3 and the vertices of a regular m-gon

$$V_j := \left(\cos(j\frac{2\pi}{m}), \sin(j\frac{2\pi}{m})\right), \; j = 0, 1, \dots, m \tag{3.48}$$

as the initial control points. The characteristic polynomial $\mathbf{a}_3^{(\mathbf{k})}(z)$ now becomes

$$\begin{aligned} \mathbf{a}_3^{(\mathbf{k})}(z) &= \frac{2^{-2}}{\cos(\frac{l}{2^{k+1}})\cos(\frac{l}{2^k})}(1+e^{-i\frac{l}{2^k}}z)(1+e^{i\frac{l}{2^k}}z)(1+z) \\ &= \frac{\cos(\frac{l}{2^{k+1}})}{\cos(\frac{l}{2^k})}\frac{2^{-2}}{\cos^2(\frac{l}{2^{k+1}})}(1+e^{-i\frac{l}{2^k}}z)(1+e^{i\frac{l}{2^k}}z)(1+z). \end{aligned} \tag{3.49}$$

By Example 1.2 if the points V_j, $j = 0, 1, \dots, m$ are taken as the initial control points, then the limit curve of the BSS associated with the characteristic polynomial

$$\mathbf{c}_3^{(\mathbf{k})}(z) = \frac{2^{-2}}{\cos^2(\frac{l}{2^{k+1}})}(1+e^{-i\frac{l}{2^k}}z)(1+e^{i\frac{l}{2^k}}z)(1+z), \; l = 2\pi/m$$

is the circle with origin $(0, 0)$ and radius $\cos(2\pi/m)$ inscribed in that regular m-gon whose vertices are given by V_j, $j = 0, 1, \dots, m$.

Note that

$$\lim_{k\to\infty}\prod_{j=0}^{k}\frac{\cos(\frac{l}{2^{j+1}})}{\cos(\frac{l}{2^j})} = \frac{1}{\cos(l)}.$$

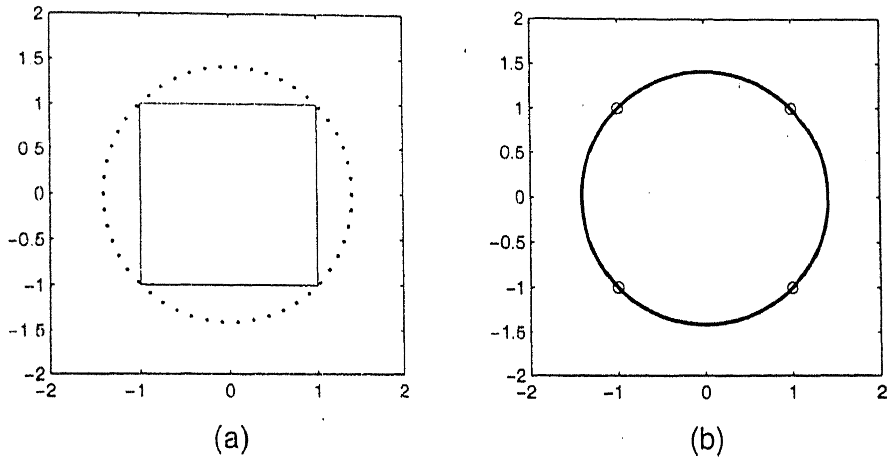


Figure 3.3: (a) The square whose vertices lie equidistantly on the unit circle (dotted curve) is considered as initial control polygon, (b) Limit curve after third iteration

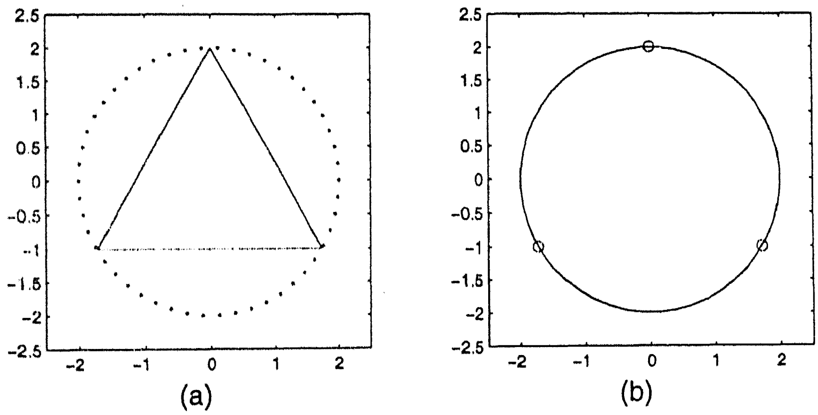


Figure 3.4: (a) The regular triangle whose vertices lie equidstantly on the unit circle (dotted curve) is taken as initial control polygon (b) Limit curve after third iteration

Therefore when $l = \frac{2\pi}{m}$ the limit curve of the BSS associated with the polynomial $\mathbf{a}_3^{(\mathbf{k})}(z)$ is the circle with origin $(0,0)$ and radius 1. Since the distance of origin $(0,0)$ to V_j is 1 the resulting circle passes through the vertices V_j, $j = 0, 1, \ldots, m$.

Chapter 4

Interpolatory Scheme

The chapter is organized as follows. In Section 4.1 the subdivision scheme is introduced and the convergence of the scheme is also studied. Some properties of the basic limit function are presented in Section 4.2. In Section 4.3 we show that the scheme reconstructs a certain class of trigonometric polynomials. The quadratic order of approximation of the limit function is shown in Section 4.4. The scheme is used to interpolate a complex sequence in Section 4.5.

4.1 The Subdivision Algorithm and its Convergence

Let us define a space of trigonometric polynomials $\mathcal{T}$ by

$$\mathcal{T} := span\{1, c(x), s(x), c(2x), s(2x)\}. \tag{4.1}$$

where $c(x) = \cos(\alpha x)$ and $s(x) = \sin(\alpha x)$, $0 \le \alpha < \pi$.

Let $g \in \mathcal{T}$ and $g(x) = a_0 + a_1 c(x) + a_2 s(x) + a_3 c(2x) + a_4 s(2x)$. Then the sum $a_3^2 + a_4^2$ is called the amplitude of g.

Suppose we have a data set

$$D = \{(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))\}.$$

Then there exist several functions in $\mathcal{T}$ interpolating the data set D. But it is known [53] that the function

$$L(x) = \sum_{j=0}^{3} f(x_j)L_j(x) \tag{4.2}$$

where

$$L_j(x) = c(\frac{x-x_j}{2}) \prod_{k=0,\, k\neq j}^{3} \frac{s(\frac{x-x_k}{2})}{s(\frac{x_j-x_k}{2})} \tag{4.3}$$

is the unique function in $\mathcal{T}$, which interpolates D and has the minimum amplitude among other interpolants from $\mathcal{T}$. We call the function $L(x)$ a **Lagrange like interpolant** of the above data.

Define $x_j = j$, $j = 0, 1, 2, 3$ and $x' = 3/2$ then

$$L_0(x') = L_3(x') = -\frac{1}{2}\frac{\sin^2(\alpha/4)}{\sin(\alpha/2)\,\sin(\alpha)}, \tag{4.4}$$

$$L_1(x') = L_2(x') = \frac{1}{2}\frac{\sin^2(3\,\alpha/4)}{\sin(\alpha/2)\,\sin(\alpha)}. \tag{4.5}$$

Clearly,

$$\frac{\sin^2(3\alpha/4)}{\sin(\alpha/2)\,\sin(\alpha)} - \frac{\sin^2(\alpha/4)}{\sin(\alpha/2)\,\sin(\alpha)} = 1. \tag{4.6}$$

Let $w_0 = \frac{\sin^2(\alpha/4)}{2\sin(\alpha/2)\sin(\alpha)}$. Then $\frac{1}{2} + w_0 = \frac{\sin^2(3\alpha/4)}{2\sin(\alpha/2)\sin(\alpha)}$ and

$$L(x') = -w_0(f(x_0) + f(x_3)) + (\frac{1}{2} + w_0)(f(x_1) + f(x_2)).$$

To define our nonstationary scheme, for $k \geq 0$, we denote

$$w_k = \frac{\sin^2(\frac{\alpha}{2^{k+2}})}{2\,\sin(\frac{\alpha}{2^k})\,\sin(\frac{\alpha}{2^{k+1}})} = \frac{1}{16\,\cos^2(\frac{\alpha}{2^{k+2}})\,\cos(\frac{\alpha}{2^{k+1}})}. \tag{4.7}$$

Some estimates of w_k which are useful in our scheme are given in the following lemma.

Lemma 4.1. *For $k \geq 0$ and $0 \leq \alpha \leq \pi/2$*

(a) $\frac{1}{8} \geq w_k \geq \frac{1}{16}$

(b) $|w_k - \frac{1}{16}| \leq \frac{C}{2^{2k}}$ for some constant C independent of k.

Proof: The inequality $w_k \geq \frac{1}{16}$ is obvious from (4.7). Observe that

$$w_k = \frac{1}{16\,\cos^2(\frac{\alpha}{2^{k+2}})\,\cos(\frac{\alpha}{2^{k+1}})} \leq \frac{1}{8\left(\cos^2(\frac{\alpha}{2^{k+1}}) + \cos(\frac{\alpha}{2^{k+1}})\right)} \leq \frac{1}{8}$$

Below we establish the asymptotic equivalence of $\{S_k\}$ and $\{S\}$.

Theorem 4.3. *The nonstationary scheme $\{S_k\}$ is asymptotically equivalent to the stationary scheme $\{S\}$. Moreover, the limit function belongs to $C^1(\mathbb{R})$.*

Proof: We have

$$\sum_{\beta\in\mathbb{Z}} |a^{(k)}_{2\beta} - a_{2\beta}| = 0 \quad \text{and} \quad \sum_{\beta\in\mathbb{Z}} |a^{(k)}_{1+2\beta} - a_{1+2\beta}| = 4|w_k - 1/16|.$$

By Lemma 4.1(b) we get $\sum_{\beta\in\mathbb{Z}} |a^{(k)}_{1+2\beta} - a_{1+2\beta}| \leq \frac{4C}{2^{2k}}$ and hence

$$\left\|S_k - S\right\| \leq \frac{4\,C}{2^{2k}}. \tag{4.11}$$

Hence $\sum_{k=0}^{\infty} \|S_k - S\|_\infty < \infty$ and the schemes $\{S_k\}$ and $\{S\}$ are asymptotically equivalent. Since $\{S\}$ is a C^0 stationary scheme, by Theorem 1.2 the scheme $\{S_k\}$ is a stable and C^0 convergent scheme. Moreover, from (4.11) it is clear that

$$\sum_{k=0}^{\infty} 2^k \left\|S_k - S\right\|_\infty < \infty. \tag{4.12}$$

As the scheme associated with S converges to a C^1 limit function, by Theorem 1.3 the scheme $\{S_k\}$ also converges to a C^1 limit function. This proves the theorem. □

We illustrate the subdivision scheme in Figure 1. Initially we take a finite set of points (Figure 1(a)). The limit curves after three iterations of our algorithms are shown in Figure 1(b)-1(d).

4.2 Basic Limit Function

The basic limit function of the scheme $\{S_k\}$ is the limit function of the scheme for the data

$$p_i^0 = \begin{cases} 1 & i = 0, \\ 0 & i \neq 0. \end{cases} \tag{4.13}$$

since $\cos^2(x) + \cos(x) > 1 \; \forall x \in [0, \pi/4]$. This proves (a). Also note that

$$w_k - \frac{1}{16} = \frac{1}{16}\left(\frac{1-\cos^2(\frac{\alpha}{2^{k+2}})\cos(\frac{\alpha}{2^{k+1}})}{\cos^2(\frac{\alpha}{2^{k+2}})\cos(\frac{\alpha}{2^{k+1}})}\right)$$
$$= \frac{\left(2+\cos(\frac{\alpha}{2^{k+1}})\right)\sin^2(\frac{\alpha}{2^{k+2}})}{16\cos^2(\frac{\alpha}{2^{k+2}})\cos(\frac{\alpha}{2^{k+1}})} < \frac{3\alpha^2}{16\cos^2(\alpha/4)\cos(\alpha/2)}\frac{1}{2^{2k+4}}.$$

The lemma follows by choosing $C = \alpha^2/\big(\cos^2(\alpha/4)\cos(\alpha/2)\big)$. □

Now, we present the basic algorithm which is a nonstationary subdivision scheme.

Algorithm 4.1. *Given the control points $\{p_i^0 \in \mathbb{R},\ i = -2, -1, \ldots, n+2\}$ the control points $\{p_i^{k+1},\ i = -2, -1, 0, \ldots, 2^{k+1}n+1\}$ at level $k+1$ are given by the following recursive relation:*

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \quad -1 \le i \le 2^k n + 1 \\ p_{2i+1}^{k+1} &= -w_k p_{i-1}^k + (\frac{1}{2} + w_k)p_i^k + (\frac{1}{2} + w_k)p_{i+1}^k - w_k p_{i+2}^k, \quad -1 \le i \le 2^k n. \end{aligned} \tag{4.8}$$

Remark 4.2. If we take $w_k = \frac{1}{16}$ for all k, then this scheme coincides with Dyn and Levin's four point subdivision scheme [32]. The point $(\frac{2i+1}{2^{k+1}}, p_{2i+1}^{k+1})$ is lying on the Lagrange like interpolant of the points

$$\{(\frac{i-1}{2^k}, p_{i-1}^k), (\frac{i}{2^k}, p_i^k), (\frac{i+1}{2^k}, p_{i+1}^k), (\frac{i+2}{2^k}, p_{i+2}^k)\}.$$

Note that the set of points at $(k+1)$th level of the algorithm contains all the points at the kth level and some new points. Therefore the initial set of control points is contained at all levels of the algorithm. Therefore, the limit curve interpolates the set of initial points.

Let us denote our non-stationary scheme by $\{S_k\}$. The mask of $\{S_k\}$ at the kth level is $a^{(k)} = \{a_{-3}^{(k)}, \ldots, a_3^{(k)}\}$ where

$$a_{-3}^{(k)} = a_3^{(k)} = -w_k,\ a_{-2}^{(k)} = a_2^{(k)} = 0,\ a_0^{(k)} = 1,\ a_{-1}^{(k)} = a_1^{(k)} = 1/2 + w_k. \tag{4.9}$$

Note that the subdivision operator S, associated with the four point scheme of Dyn and Levin has the mask $a = \{a_{-3}, \ldots, a_3\}$ where

$$a_{-3} = a_3 = -1/16,\ a_{-2} = a_2 = 0,\ a_0 = 1,\ a_{-1} = a_1 = 9/16. \tag{4.10}$$

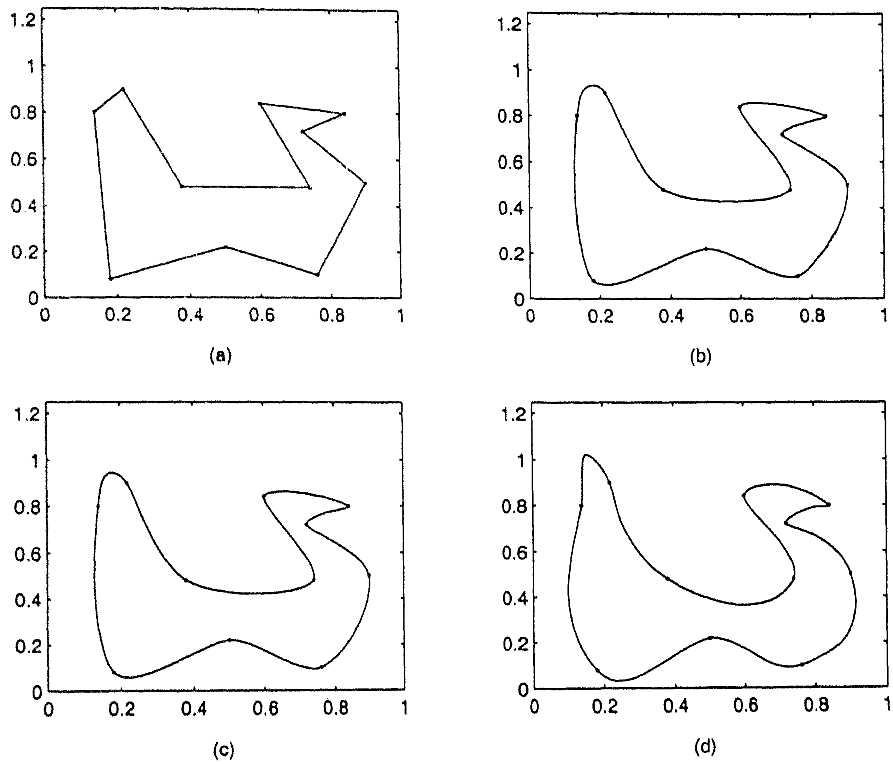


Figure 4.1: (a) Initial data. (b) Limit curve for $\alpha = 0.25$. (c) Limit curve for $\alpha = 0.5$. (d) Limit curve for $\alpha = 1.0$.

By Theorem 4.3 the basic limit function belongs to $C^1(\mathbb{R})$. In this section we derive some basic properties of the basic limit function denoted by F. Let

$$D_n := \{\frac{j}{2^n},\ j \in \mathbb{Z}\}. \tag{4.14}$$

It is easy to check that restriction of F to D_n satisfies $F(\frac{j}{2^n}) = p_j^n\ \forall j$.

Theorem 4.4. *F is symmetric about the Y-axis.*

Proof: We prove this by induction on n. First of all $F(\frac{j}{2^n}) = F(-\frac{j}{2^n})$ for $n = 0$ since $F(j) = F(-j) = 0,\ j \in \mathbb{Z}$ by interpolation property. Let us assume that $F(\frac{j}{2^k}) = F(-\frac{j}{2^k}),\ j \in \mathbb{Z},\ k = 1, 2, \ldots, n$. Therefore, $F(\frac{2j}{2^{n+1}}) = F(-\frac{2j}{2^{n+1}}),\ \forall j$. Moreover,

$$\begin{aligned}
F(\frac{2j+1}{2^{n+1}}) &= -w_n F(\frac{j-1}{2^n}) + (1/2 + w_n)F(\frac{j}{2^n}) + (1/2 + w_n)F(\frac{j+1}{2^n}) - w_n F(\frac{j+2}{2^n}) \\
&= -w_n F(\frac{-j+1}{2^n}) + (1/2 + w_n)\Big(F(\frac{-j}{2^n}) + F(\frac{-j-1}{2^n})\Big) - w_n F(\frac{-j-2}{2^n}) \\
&= F(-\frac{2j+1}{2^{n+1}}).
\end{aligned}$$

Hence $F(\frac{j}{2^n}) = F(-\frac{j}{2^n})$ for all j and $n \in \mathbb{Z}$. From the continuity of F we have $F(x) = F(-x)$ $\forall x \in \mathbb{R}$ which completes the proof of the theorem. □

The translation $F(\cdot - k)$, $k \in \mathbb{R}$ of F is the limit function of the scheme for the initial data

$$p_i^0 = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases} \tag{4.15}$$

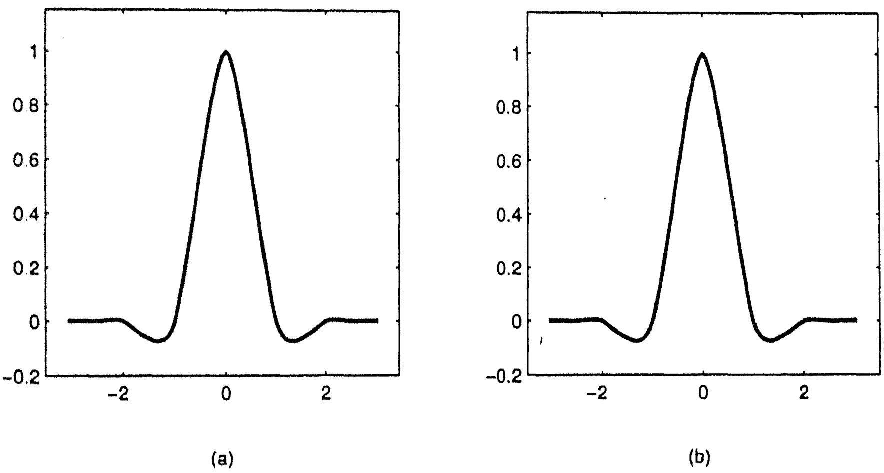


Figure 4.2: Basic limit functions: (a) for $\alpha = 0.25$ and (b) for $\alpha = 0.5$ respectively

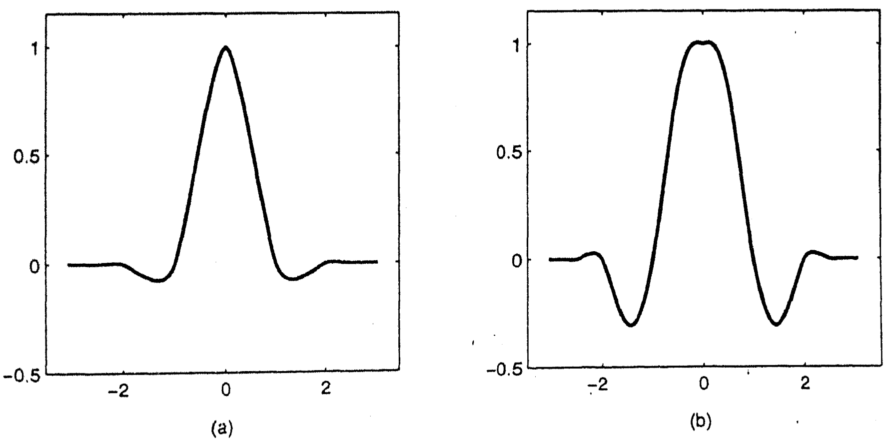


Figure 4.3: Basic limit functions: (a) for $\alpha = 1.0$ and (b) for $\alpha = 2.5$ respectively

We show that F is a compactly supported function with support in $[-3, 3]$ (See Figure 4.2-4.3).

Theorem 4.5. *The basic limit function F vanishes outside $[-3,3]$.*

Proof: Let us take $t_0 = 0$ and define t_n recursively by $t_{n+1} = t_n + \frac{3}{2^{n+1}}$. Observe that $t_n \in D_n$ but $t_n \notin D_{n-1}$. We claim that the restriction of F to D_n vanishes outside $[-t_n, t_n]$. Since F is symmetric it is enough to prove that F vanishes outside $[0, t_n]$. We prove this by induction on n.

Observe that from Remark 4.2

$$\begin{aligned} F(t_{n+1} + \frac{2k}{2^{n+1}}) &= F(t_n + \frac{2k+3}{2^{n+1}}) \\ &= -w_n\big(F(t_n + \frac{k}{2^n}) + F(t_n + \frac{k+3}{2^n})\big) \\ &\quad + (1/2 + w_n)\big(F(t_n + \frac{k+1}{2^n}) + F(t_n + \frac{k+2}{2^n})\big) \end{aligned} \tag{4.16}$$

and

$$F(t_{n+1} + \frac{2k+1}{2^{n+1}}) = F(t_n + \frac{k+2}{2^n}). \tag{4.17}$$

From the above equations it is easy to check that $F(t_1) = -w_0 F(0) \neq 0$ and for $k > 0$

$$F(t_1 + \frac{2k-1}{2}) = 0 \quad \text{and} \quad F(t_1 + \frac{2k}{2}) = 0.$$

This proves our claim for $n = 1$.

Let us assume $F(x) = 0$ whenever $x \in D_n$ and $x > t_n$. Then by (4.16) and (4.17) we get $F(t_{n+1}) = -w_n F(t_n) \neq 0$ and for $k > 0$ we have $F(t_{n+1} + \frac{2k}{2^{n+1}}) = 0$ and $F(t_{n+1} + \frac{2k-1}{2^{n+1}}) = 0$. Since $[-t_n, t_n]$ is contained in $[-3,3]$ for all n and $\lim_{n\to\infty}[-t_n, t_n] = [-3,3]$ we have the required result. $\square$

Remark: Since F is compactly supported and continuous function on R, it follows that $F(\cdot - k)$ are compactly supported continuous functions as well. If $y(t)$ is the limit function interpolating the initial sequence $\{y(n)\}$. Then clearly,

$$y(t) = \sum_{j\in\mathbb{Z}} y(j)F(t-j) = \sum_{j=k-2}^{k+3} y(j)F(t-j) \tag{4.18}$$

where k is the greatest integer such that $k \leq t$.

4.3 Reconstruction of Functions

In this section we show that certain functions can be reconstructed by our scheme. It is easy to check that if $p_i^k = 1$ for all i at kth level then $p_j^{k+1} = 1$ for all j at $k+1$ th level. This shows that the function $f(x) = 1$ is reproduced by our scheme. Another simple consequence of this fact is that

$$\sum_{k\in\mathbb{Z}} F(t-k) = 1, \quad t \in \mathbb{R}. \tag{4.19}$$

Therefore the translations of the basic limit function F form a partition of unity.

The functions $\cos(\alpha x)$ and $\sin(\alpha x)$ can also be reconstructed by our scheme which follows from the following lemma.

Lemma 4.6. *Let $k \geq 0$ and $n > 0$ be fixed integers. Let $p_j^k = \cos(j\frac{\alpha}{2^k})$, $-2 \leq j \leq 2^k n + 2$. Then, we have*

$$p_{2i}^{k+1} = \cos(\frac{2i\alpha}{2^{k+1}}) \quad \textit{and} \quad p_{2i+1}^{k+1} = \cos((2i+1)\frac{\alpha}{2^{k+1}}), \quad -1 \leq i \leq 2^k n.$$

Similarly, if $p_j^k = \sin(j\frac{\alpha}{2^k})$ then,

$$p_{2i}^{k+1} = \sin(\frac{2i\alpha}{2^{k+1}}) \quad \textit{and} \quad p_{2i+1}^{k+1} = \sin((2i+1)\frac{\alpha}{2^{k+1}}), \quad -1 \leq i \leq 2^k n.$$

Proof: We first prove the lemma for the first case: $p_i^k = \cos(i\frac{\alpha}{2^k})$. Note that

$$p_{2i}^{k+1} = p_i^k = \cos(i\frac{\alpha}{2^k}) = \cos(2i\frac{\alpha}{2^{k+1}})$$

and

$$p_{2i+1}^{k+1} = -w_k\left(p_{i-1}^k + p_{i+2}^k\right) + (1/2 + w_k)\left(p_i^k + p_{i+1}^k\right).$$

Since

$$\frac{1}{2} + w_k = \frac{\sin^2(3\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})},$$

$$p_i^k + p_{i+1}^k = 2\cos((2i+1)\frac{\alpha}{2^{k+1}})\cos(\frac{\alpha}{2^{k+1}})$$

and

$$p_{i-1}^k + p_{i+2}^k = 2\cos((2i+1)\frac{\alpha}{2^{k+1}})\cos(3\frac{\alpha}{2^{k+1}}),$$

we get

$$
\begin{aligned}
&p^{k+1}_{2i+1}\\
&= \left(\frac{-2\sin^2(\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})} \cos(3\frac{\alpha}{2^{k+1}}) + \frac{2\sin^2(3\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})} \cos(\frac{\alpha}{2^{k+1}}) \right) \cos(\frac{2i+1}{2}\frac{\alpha}{2^k})\\
&= \frac{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})}{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})} \cos(\frac{2i+1}{2}\frac{\alpha}{2^k})\\
&= \cos((2i+1)\frac{\alpha}{2^{k+1}}).
\end{aligned}
$$

Analogously, $p^{k+1}_{2i} = \sin(\frac{2i\alpha}{2^{k+1}})$ and the expressions

$$p^k_{i-1} + p^k_{i+2} = 2\sin((2i+1)\frac{\alpha}{2^{k+1}})\cos(3\frac{\alpha}{2^{k+1}})$$

and

$$p^k_i + p^k_{i+1} = 2\sin((2i+1)\frac{\alpha}{2^{k+1}})\cos(\frac{\alpha}{2^{k+1}})$$

lead to the following expressions for p^{k+1}_{2i+1}:

$$
\begin{aligned}
&p^{k+1}_{2i+1}\\
&= \left(\frac{\sin^2(\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})} \cos(3\frac{\alpha}{2^{k+1}}) + \frac{2\sin^2(3\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^{k+1}})\sin(\frac{\alpha}{2^k})} \cos(\frac{\alpha}{2^{k+1}}) \right) \sin(\frac{2i+1}{2}\frac{\alpha}{2^k})\\
&= \sin((2i+1)\frac{\alpha}{2^{k+1}}).
\end{aligned}
$$

This proves the lemma. □

Corollary 4.7. If the initial data lie on a graph of a function $f \in \mathcal{T}$ and the values of f are given on a set of equidistant points then the limit function of scheme (4.8) exactly reproduces the original function f. In particular if we choose a set of equidistant points

$$p^0_i = (\cos(k\frac{2\pi}{n}), \sin(k\frac{2\pi}{n})),\ k = 0, 1, \ldots, n$$

on a circle, and $\alpha = 2\pi/n$, then the limit curve is the original unit circle. The Figure 4.4 and 4.5, illustrate the above observation.

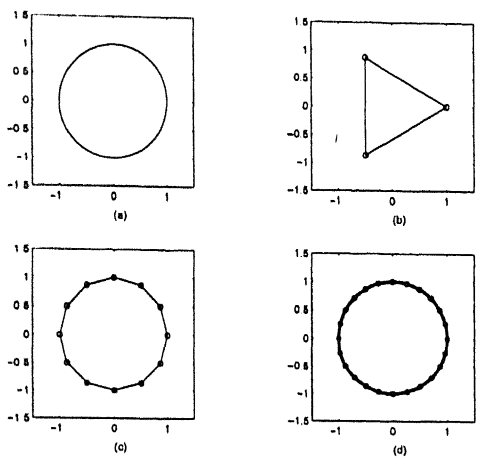


Figure 4.4: In (b) three equidistant points are taken as initial control points which lie on a circle shown in (a). The control points and the control polygons after second and third iterations are shown in Figure (c) and (d) respectively.

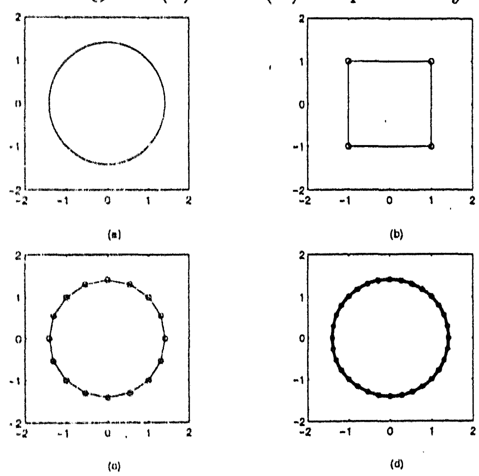


Figure 4.5: In (b) four equidistant points are taken as initial control points which lie on a circle shown in (a). The control points and the control polygons after second and third iterations are shown in Figure (c) and (d) respectively.

4.4 Order of Approximation

Finally, we present a result on the order of approximation of the above interpolation scheme. To state the basic approximation theorem, we assume that for a fixed $n \in \mathbb{N}$

let $h = \frac{1}{n}$ and $I = [-2h, 1+h]$. Suppose that g is a $C^2(I)$ function defined on I and

$$p_i^0 = g(ih), -2 \le i \le n+2.$$

The basic approximation theorem states that the limit function obtained by the above nonstationary scheme with the data $g(ih)$, $-2 \le i \le n+2$ approximates $g(t)$ with an error of $O(h^2)$. Now, we state the theorem precisely.

Theorem 4.8. *Let $p_i^0 = g(ih)$, $-2 \le i \le n+2$, $h = 1/n$. Let f be the limit function of the nonstationary interpolatory scheme (4.8). If $g \in C^2(I)$ then there exists a constant C such that*

$$\|f-g\|_{\infty,[0,1]} := \max_{0 \le x \le 1} |f(x) - g(x)| \le C \frac{h^2}{\cos(5\alpha h/2)}. \tag{4.20}$$

Proof: Let us define the compactly supported functions B_i, $i \in \mathbb{Z}$ by

$$B_i(x) := F(\frac{x}{h} - i), \quad i \in \mathbb{Z},$$

where F is the basic limit function of the nonstationary scheme (4.8). By (4.19)

$$\sum_{i \in \mathbb{Z}} B_i(x) = 1, \quad \forall x \in I. \tag{4.21}$$

Let us denote the interval $[ih, jh]$ by $I_{i,j}$. Then for $x \in I_{k,k+1}$ by (4.18) we have

$$f(x) = \sum_{i=k-2}^{k+3} p_i^0 B_i(x) = \sum_{i=k-2}^{k+3} g(ih) B_i(x). \tag{4.22}$$

Let us define

$$Q(x) = \frac{\sin(\alpha((k+3)h - x))}{\sin(5\alpha h)} p_{k-2}^0 + \frac{\sin(\alpha(x-(k-2)h))}{\sin(5\alpha h)} p_{k+3}^0.$$

It is easy to check that $Q \in \mathcal{T}$ and hence by Corollary 4.7 and (4.18) for $x \in I_{k-2,k+3}$ we get

$$Q(x) = \sum_{i=k-2}^{k+3} Q(ih) F(\frac{x}{h} - i). \tag{4.23}$$

It has been shown in [48] that if $g \in C^2(I)$ then there exists a constant C_1 depending only upon g such that

$$\|Q - g\|_{\infty, I_{k-2,k+3}} \le C_1 \frac{h^2}{\cos(5\alpha/2)}. \tag{4.24}$$

Moreover by (4.23) we get

$$\|f-Q\|_{\infty,I_{k,k+1}} = \|\sum_{j=k-2}^{k+3} g(jh)B_j(x) - \sum_{j=k-2}^{k+3} Q(jh)B_j(x)\|_{\infty,I_{k,k+1}}$$
$$\leq 5 \max_{k-2\leq j\leq k+3}\|B_j(x)\|_{\infty,I_{k-2,k+3}}\|Q-g\|_{\infty,I_{k-2,k+3}}.$$

Since the basic limit functions $B_j(x)$, $j \in \mathbb{Z}$ are bounded functions. We have

$$\|f-g\|_{\infty,I_{k,k+1}} \leq \|f-Q\|_{\infty,I_{k,k+1}} + \|Q-g\|_{\infty,I_{k,k+1}}$$
$$\leq \max\Big\{5\|B_j(x)\|_{\infty,I_{k,k+1}} + 1\Big\}C_1\frac{h^2}{\cos(5\alpha h/2)}. \tag{4.25}$$

This completes the proof of the theorem. □

4.5 Complex Interpolation

As an illustration of the our algorithm, we study the following problem of complex interpolation. Let $z = re^{ib}$ be a complex number different from zero. Let us define a complex sequence $\{z(n)\}$ by $z(n) = z^{n\alpha}\ \forall n$. Let $x(n) = Re(z(n))$ and $y(n) = Im(z(n))$ $\forall n$. Let $x(t)$ and $y(t)$ be the C^1 limit functions of the scheme $\{S_k\}$ interpolating the initial sequences $\{x(n)\}$ and $\{y(n)\}$ respectively. Then the function $z(t) = x(t) + iy(t)$, $t \in \mathbb{R}$ is called the complex interpolation of the sequence $\{z(n)\}$.

Theorem 4.9. *Let $z(t)$ be the complex interpolation of the sequence $\{z(n)\}$. Then $z(t)$ has the Fourier series expansion*

$$z(t) = \sum_{n=-\infty}^{\infty} c_n r^t e^{i(b+\frac{2\pi n}{\alpha})t}$$

where $c_n = \prod_{k=1}^{\infty} R_k(z_{k,n})$, $z_{k,n} = r^{\frac{\alpha}{2^k}} e^{i(b+\frac{2\pi n}{\alpha})\frac{\alpha}{2^k}}$ *and*

$$R_k(z) = \Big(-\frac{w_{k-1}}{z^3} + \frac{1/2+w_{k-1}}{z} + 1 + (1/2+w_{k-1})z - w_{k-1}z^3\Big)/2. \tag{4.26}$$

Proof: Let us consider the function $f(t) = \frac{z(t)}{r^t e^{ibt}}$. Then $f(t+\alpha) = \frac{z(t+\alpha)}{r^{t+\alpha}e^{ib(t+\alpha)}}$. Since $z(t+\alpha)$ is the interpolation of the sequence $\{z(n\alpha+\alpha)\}$, $z(n\alpha+\alpha) = z^{(n+1)\alpha} = r^\alpha e^{ib\alpha} z^{n\alpha}$ we have $z(t+\alpha) = r^\alpha e^{ib\alpha} z(t)$. Thus

$$f(t+\alpha) = \frac{r^\alpha e^{ib\alpha} z(t)}{r^\alpha e^{ib\alpha} r^t e^{ibt}} = f(t).$$

This implies that f is α periodic. Since $z'(t)$ is continuous, $f'(t)$ is continuous and hence

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi}{\alpha}kt} \tag{4.27}$$

where $c_k = \frac{1}{\alpha}\int_0^\alpha f(y)e^{-iky\frac{2\pi}{\alpha}}dy$.

Define $a_k = \frac{f(0)+f(\frac{\alpha}{2^k})+\cdots+f(\frac{2^k-1}{2^k}\alpha)}{2^k}$. It is easy to check that $a_0 = f(0) = z(0) = 1$ and

$$c_0 = \frac{1}{\alpha}\int_0^\alpha f(x)dx = \lim_{k\to\infty}\frac{f(0)+f(\frac{\alpha}{2^k})+\cdots+f(\frac{2^k-1}{2^k}\alpha)}{2^k} = \lim_{k\to\infty} a_k.$$

Now

$$\begin{aligned} a_k &= \frac{f(0)+f(2\frac{\alpha}{2^k})+\cdots+f(\frac{2^k-2}{2^k}\alpha)}{2^k} + \frac{f(\frac{\alpha}{2^k})+f(3\frac{\alpha}{2^k})+\cdots+f(\frac{2^k-1}{2^k}\alpha)}{2^k} \\ &= \frac{a_{k-1}}{2} + \frac{f(\frac{\alpha}{2^k})+f(3\frac{\alpha}{2^k})+\cdots+f(\frac{2^k-1}{2^k}\alpha)}{2^k}. \end{aligned}$$

Since $f(p\frac{\alpha}{2^k}) = \frac{z(p\frac{\alpha}{2^k})}{(re^{ib})^{p\frac{\alpha}{2^k}}}$ when p is odd we get

$$\begin{aligned} z(p\frac{\alpha}{2^k}) &= -w_{k-1}z((p-3)\frac{\alpha}{2^k}) + (1/2+w_{k-1})z((p-1)\frac{\alpha}{2^k}) \\ &\quad + (1/2+w_{k-1})z((p+1)\frac{\alpha}{2^k}) - w_{k-1}z((p+3)\frac{\alpha}{2^k}) \end{aligned}$$

and

$$\begin{aligned} f(p\frac{\alpha}{2^k}) &= -\frac{1}{\gamma^3}w_{k-1}f((p-3)\frac{\alpha}{2^k}) + \frac{1}{\gamma}(1/2+w_{k-1})f((p-1)\frac{\alpha}{2^k}) \\ &\quad + \gamma(1/2+w_{k-1})f((p+1)\frac{\alpha}{2^k}) - \gamma^3 w_{k-1}f((p+3)\frac{\alpha}{2^k}) \end{aligned}$$

where $\gamma = r^{\frac{\alpha}{2^k}}e^{ib\frac{\alpha}{2^k}}$. Therefore

$$\begin{aligned} &\frac{f(\frac{\alpha}{2^k})+f(3\frac{\alpha}{2^k})+\cdots+f(\frac{2^k-1}{2^k}\alpha)}{2^k} \\ &= \frac{1}{2^k}\Big(\frac{-w_{k-1}}{\gamma^3}f(-\frac{\alpha}{2^{k-1}}) + \frac{1/2+w_{k-1}}{\gamma}f(0) + (1/2+w_{k-1})f(\frac{\alpha}{2^{k-1}})\gamma - w_{k-1}f(2\frac{\alpha}{2^{k-1}})\gamma^3 \\ &\quad - \frac{w_{k-1}}{\gamma^3}f(0) + \frac{1/2+w_{k-1}}{\gamma}f(\frac{\alpha}{2^{k-1}}) + (1/2+w_{k-1})f(2\frac{\alpha}{2^{k-1}})\gamma - w_{k-1}f(3\frac{\alpha}{2^{k-1}})\gamma^3 \\ &\quad - \quad \vdots \quad + \quad \vdots \quad + \quad \vdots \quad - \quad \vdots \\ &\quad - \frac{w_{k-1}}{\gamma^3}f(\frac{2^{k-1}-2}{2^{k-1}}\alpha) + \frac{1/2+w_{k-1}}{\gamma}f(\frac{2^{k-1}-1}{2^{k-1}}\alpha) + (1/2+w_{k-1})f(\alpha)\gamma \end{aligned}$$

$$- w_{k-1} f(\frac{2^{k-1}+1}{2^{k-1}}\alpha)\gamma^3\Big).$$

Using periodicity of f we get

$$a_k = \frac{a_{k-1}}{2} + \frac{a_{k-1}}{2}\{-\frac{w_{k-1}}{\gamma^3} + \frac{1/2 + w_{k-1}}{\gamma} + (1/2 + w_{k-1})\gamma - w_{k-1}\gamma^3\}.$$

Which implies

$$a_k = a_{k-1} R_k(z_{k,0}) = \cdots = \prod_{j=1}^{k} R_j(z_{j,0}).$$

Hence $c_0 = \lim_{k\to\infty} a_k = \prod_{j=1}^{\infty} R_j(z_{j,0})$. In order to evaluate c_k, $k \geq 1$ consider the function

$$g(t) = e^{-i\frac{2\pi}{\alpha}kt} f(t) = \sum_{j=-\infty}^{\infty} c_{k+j} e^{i\frac{2\pi}{\alpha}jt} = \sum_{j=-\infty}^{\infty} c'_j e^{i\frac{2\pi}{\alpha}jt}.$$

Note that $g(t) = \frac{z(t)}{r^t e^{i(b+\frac{2\pi}{\alpha}k)t}} = \frac{z(t)}{r^t e^{imt}}$. where $m = b + \frac{2\pi}{\alpha}k$. Therefore by following the same procedure used to find c_0 we get $c_k = c'_0 = \prod_{j-1}^{\infty} R_j(z'_{j,0})$. Since $z'_{j,0} = z_{j,k}$ the theorem follows. $\square$

Chapter 5

Design of Surfaces

A non-stationary subdivision scheme for generating surfaces from arbitrary topologies is introduced. The limit surface is a bi-quadratic tensor product trigonometric spline surface except at a small number of points termed as extraordinary points. The chapter is organized into nine sections. In Section 5.1 the topology of a data set is defined. The subdivision scheme for a tensor product topology is introduced in Section 5.2. The subdivision scheme for an arbitrary topology is introduced in Section 5.3. The parameterization and prolongation procedures associated with the scheme are discussed in Section 5.4 and Section 5.5 respectively. The associated subdivision matrix is analyzed in Section 5.6. Convergence of the scheme and continuity of the tangent plane of the limit surface are shown in Section 5.7 and Section 5.8 respectively. Finally, some illustrative examples are presented in Section 5.9.

5.1 Topology of a Data Set

For designing a surface initially we are given a finite set of points. Usually the points are considered as vertices of a polyhedron. The lines joining the control points constitute edges and faces of the polyhedron.

Order of the face: The number of edges consistuting the face.

Order of the vertex The number of the edges emanating from the vertex.

A polyhedron is said to have a tensor product topology if all its faces and vertices have order 4. If the order of the faces and vertices are arbitrary, the polyhedron is said to have an arbitrary topology. A data set have the topology of the corresponding polyhedron.

5.2 Scheme for Tensor Product Topology

We recall that in Chapter 3 a trigonometric spline curve is generated by a binary subdivision scheme (BSS). Usually a finite set of control points $\{p_i^0 \in \mathbb{R},\ i \in \mathbf{Z}\}$ is given. Then the BSS for a quadratic trigonometric spline recursively generates new control points which at k-th level are obtained by the rule:

$$\begin{aligned} p_{2i}^k &= w_0\, p_{i-1}^{k-1} + w_1\, p_i^{k-1} \\ p_{2i+1}^k &= w_1\, p_{i-1}^{k-1} + w_0\, p_i^{k-1} \end{aligned} \tag{5.1}$$

where $w_0 = \frac{\sin(h/2^k)}{\sin(4h/2^k)}$ and $w_1 = \frac{\sin(3h/2^k)}{\sin(4h/2^k)}$.

The above BSS is generalized to a tensor product BSS as follows: Let

$$P^0 = \{p_{ij}^0 \in \mathbb{R}^3,\ i, j \in \mathbb{Z}\}$$

denote the initial set of control points which has a tensor product topology. These are aslo called control points at the 0-th level. Then the tensor product BSS defines recursively the k-th level control points

$$\{p_{ij}^k \in \mathbb{R}^3,\ i, j \in \mathbb{Z}\}$$

following the rule:

$$\begin{aligned} p_{2i,2j}^k &= w_1 \left\{w_1\, p_{i-1,j-1}^{k-1} + w_0\, p_{i,j-1}^{k-1}\right\} + w_0 \left\{w_1\, p_{i-1,j}^{k-1} + w_0\, p_{i,j}^{k-1}\right\} \\ p_{2i+1,2j}^k &= w_1 \left\{w_0\, p_{i-1,j-1}^{k-1} + w_1\, p_{i,j-1}^{k-1}\right\} + w_0 \left\{w_0\, p_{i-1,j}^{k-1} + w_1\, p_{i,j}^{k-1}\right\} \\ p_{2i,2j+1}^k &= w_0 \left\{w_1\, p_{i-1,j-1}^{k-1} + w_0\, p_{i,j-1}^{k-1}\right\} + w_1 \left\{w_1 p_{i-1,j}^{k-1} + w_0\, p_{i,j}^{k-1}\right\} \\ p_{2i+1,2j+1}^k &= w_0 \left\{w_0\, p_{i-1,j-1}^{k-1} + w_1\, p_{i,j-1}^{k-1}\right\} + w_1 \left\{w_0\, p_{i-1,j}^{k-1} + w_1\, p_{i,j}^{k-1}\right\}. \end{aligned} \tag{5.2}$$

where

$$M = \begin{pmatrix} \frac{\sin(\frac{3h}{2})}{\sin(2h)} & \frac{\sin(\frac{h}{2})}{\sin(2h)} & 0 \\ \frac{\sin(\frac{h}{2})}{\sin(2h)} & \frac{\sin(\frac{3h}{2})}{\sin(2h)} & 0 \\ 0 & \frac{\sin(\frac{3h}{2})}{\sin(2h)} & \frac{\sin(\frac{h}{2})}{\sin(2h)} \end{pmatrix}. \tag{5.4}$$

The superscript t in (5.3) denotes the matrix transpose.

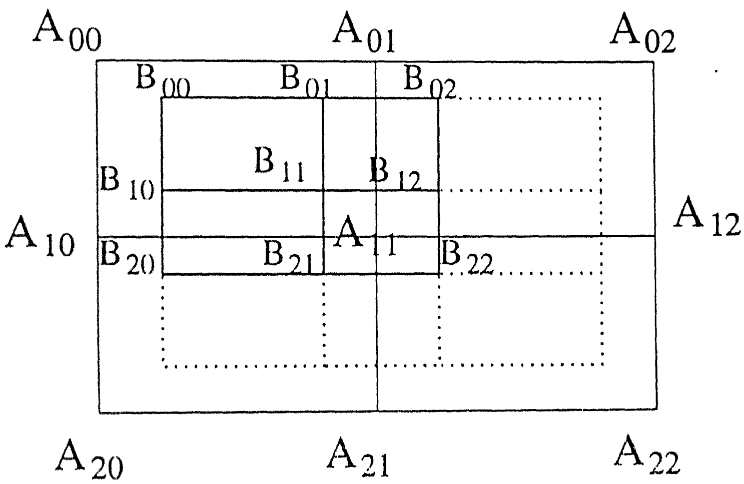


Figure 5.2: Old and new control points

After simplification we have

$$B_{00} = \frac{P + E_1 + E_2 + F}{4} \tag{5.5}$$

where

$$P = \frac{1}{\cos^2(\frac{h}{2})} A_{00}, \tag{5.6}$$

$$E_1 = \frac{1}{\cos^2(h/2)\cos(h)} \frac{A_{00} + A_{01}}{2}, \tag{5.7}$$

$$E_2 = \frac{1}{\cos^2(h/2)\cos(h)} \frac{A_{00} + A_{10}}{2}, \tag{5.8}$$

and

$$F = \frac{1}{\cos^2(h)\cos^2(h/2)} \frac{A_{00} + A_{01} + A_{10} + A_{11}}{4}. \tag{5.9}$$

The application of tensor product schemes always requires that the data points lie on a tensor product topology, consequently, the smoothness properties of limit surfaces are consequences of similar results for the univariate case (see Chapter 3).

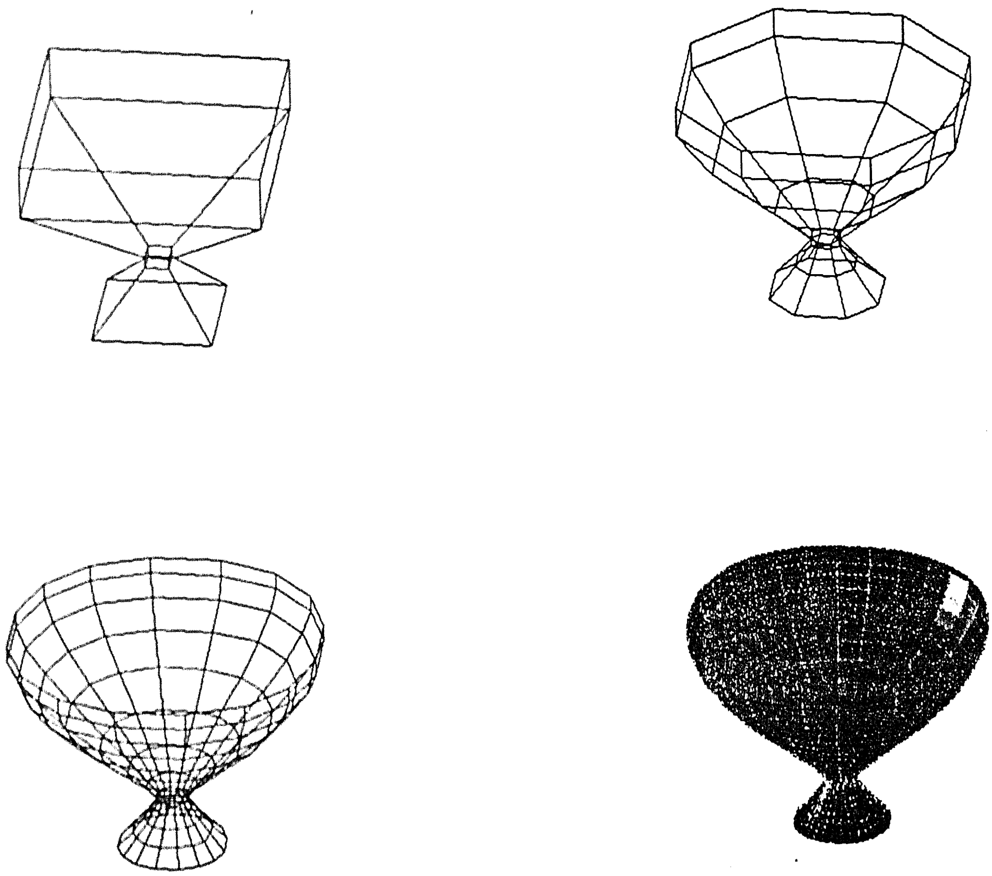


Figure 5.1: Original polyhedron and limit surfaces after first, second and third iteration

The above Figure illustrates the scheme (5.2).

Since a minimum of three control points are required to get a quadratic trigonometric spline curve, a minimum of nine control points with a tensor product topology is required to get a bi-quadratic tensor product trigonometric spline patch. Let us take such a set of nine control points (see Figure 5.2) and denote them by A_{ij}, $i, j = 0, 1, 2$.

Using the Rule (5.2) four new control points corresponding to each face and total sixteen new control points are obtained after the first iteration. Let us denote the new set of control points lying in the first three row and first three column by $B_{i,j}$, $i, j = 0, 1, 2$ (see Figure 5.2). This set of new control points has a matrix relation

$$\begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix} = M \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} M^t \tag{5.3}$$

Quite often the control points lie on an arbitrary topology. The purpose of present investigation is to generalize the above tensor product BSS in order to generate a surface from the control points lying on an arbitrary topology.

5.3 Subdivision Scheme for Arbitrary Topology

Let $\mathcal{F}^0$ be the initial control polyhedron with an arbitrary topology consisting of planar faces which are convex. The subdivision scheme defines the new control polyhedron $\mathcal{F}^1$ by the following procedure.

Let F^0 be a face of $\mathcal{F}^0$ with vertices $A_1, A_2, \ldots, A_n$ (Figure 5.3). A set of new vertices associated with the face F^0 and the old vertices A_is is obtained by the **subdivision rule:**

$$\mathbf{R1}: \quad a_i = \frac{P + E_i + E_{i-1} + F}{4} \tag{5.10}$$

where

$$P = \frac{A_i}{\cos^2(h/2)}, \quad E_i = \frac{A_i + A_{i+1}}{2\cos^2(h/2)\cos(h)}$$

and

$$F = \frac{1}{\cos^2(h)\cos^2(h/2)} \frac{A_1 + \ldots + A_n}{n}$$

where all the indices are taken modulo n. After simplification we obtain

$$a_i = \alpha(n,1)\, A_i + \beta(n,1)\, A_{i-1} + \beta(n,1)\, A_{i+1} + \gamma(n,1)\,(A_{i+2} + \ldots + A_{i+n-2}) \tag{5.11}$$

where

$$\begin{aligned}
\gamma(n,1) &= \frac{1}{4n}\frac{1}{\cos^2(\frac{h}{2})\cos^2(h)} \\
\beta(n,1) &= \gamma(n,1) + \frac{1}{8\cos^2(\frac{h}{2})\cos(h)} \\
\alpha(n,1) &= \gamma(n,1) + \frac{1}{4\cos^2(\frac{h}{2})\cos(h)} + \frac{1}{4\cos^2(\frac{h}{2})}.
\end{aligned} \tag{5.12}$$

Once all the new vertices associated with all the faces of $\mathcal{F}^0$ are obtained the following **connectivity rule** is applied to join them:

(A) For a face F of $\mathcal{F}^0$, join the new vertices associated with the old vertices in the same order as the corresponding old vertices.

(B) Consider any old vertex of $\mathcal{F}^0$ and the new vertices associated to it. Join each pair of new vertices across the adjacent faces of $\mathcal{F}^0$.

(C) Consider two adjacent faces F_1 and F_2 in $\mathcal{F}^0$. Let P_1 and P_2 be two new vertices associated with F_1 and F_2 respectively and associated with the same vertex of $\mathcal{F}^0$. Join P_1 and P_2.

As a consequence new faces enclosed by new sets of edges are obtained. Thus a new polyhedron $\mathcal{F}^1$ consisting of these new faces is generated (see Figure 5.4). We denote the whole process by an operator S_1, i.e.

$$\mathcal{F}^1 = S_1 \mathcal{F}^0.$$

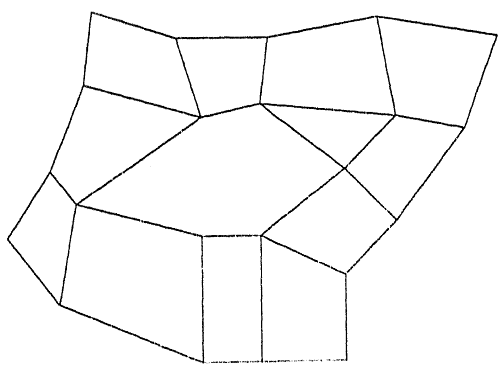


Figure 5.3: Original control polyhedron, $\mathcal{F}^0$.

Analogously, another polyhedron $\mathcal{F}^2$ (Figure 5.5) is generated from $\mathcal{F}^1$ using the subdivision rule **R2**, where the subdivision rule **Rk**, $k > 1$ is same as the subdivision rule **R1** but for the replacement of $\alpha(n,1)$, $\beta(n,1)$ and $\gamma(n,1)$ by $\alpha(n,k)$, $\beta(n,k)$ and $\gamma(n,k)$ respectively, where for $k \geq 1$

$$\begin{aligned}
\gamma(n,k) &= \frac{1}{4n} \frac{1}{\cos^2(\frac{h}{2^k}) \cos^2(\frac{h}{2^{k-1}})} \\
\beta(n,k) &= \gamma(n,k) + \frac{1}{8\cos^2(\frac{h}{2^k}) \cos(\frac{h}{2^{k-1}})} \\
\alpha(n,k) &= \gamma(n,k) + \frac{1}{4\cos^2(\frac{h}{2^k}) \cos(\frac{h}{2^{k-1}})} + \frac{1}{4\cos^2(\frac{h}{2^k})}.
\end{aligned} \tag{5.13}$$

Here $0 \leq h < \frac{\pi}{3}$. In general, the k-th control polyhedron $\mathcal{F}^k$ is generated from $\mathcal{F}^{k-1}$ using the subdivision rule **Rk**. This process is expressed by

$$\mathcal{F}^k = S_k \mathcal{F}^{k-1} = S_k S_{k-1} \cdots S_1 \mathcal{F}^0 := S_1^{(k)} \mathcal{F}^0.$$

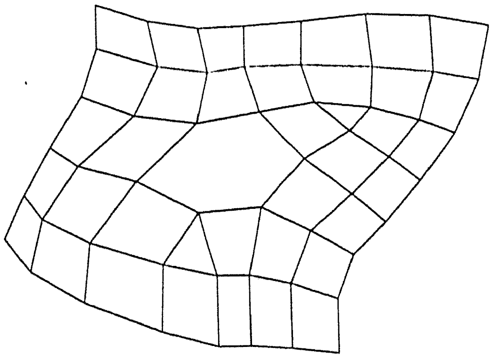


Figure 5.4: $\mathcal{F}^1$, the control polyhedron after first iteration

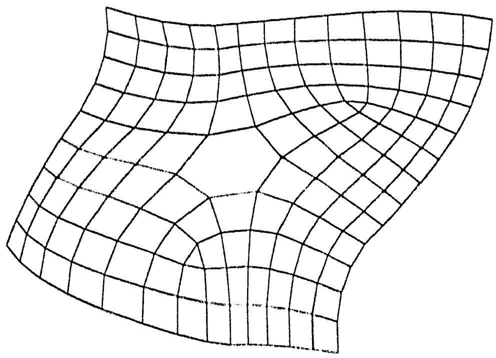


Figure 5.5: $\mathcal{F}^2$, the control polyhedron after second iteration

Our aim is to study the convergence of the sequence $\{\mathcal{F}^k\}$ to a smooth surface. First of all note that the connectivity rule ensures that a face of $\mathcal{F}^0$ of order m gives rise to a new face of order m; an interior vertex of $\mathcal{F}^0$ of order m generates a new face of order m sorrounding it and each interior edge of $\mathcal{F}^0$ gives rise to a new quadrilateral. Moreover, all the interior vertices in $\mathcal{F}^1$ will have order 4. Therefore, while the number of quadrilaterals increases, number of non-quadrilaterals remains unaltered after each iteration S_k, $k \geq 2$. However, after each iteration the sizes of these faces decrease. Besides, the quadrilaterals sorround the interior non-quadrilaterals and the number of these quadrilaterals increases as we progress with the iterations.

Following the discussion of Section 5.2, the layers of quadrilaterals sorrounding the non-quadrilaterals obtained by repeated use of subdivision rule and connectivity rule

converge to a tensor product biquadratic spline surface, while the non-quadrilaterals shrink to points termed as **extraordinary points** [26]. The above surface is C^1 except possibly at a small number of extraordinary points. In the following sections, our main concern shall be the study of the continuity and smoothness of the above limit surfaces at these extraordinary points.

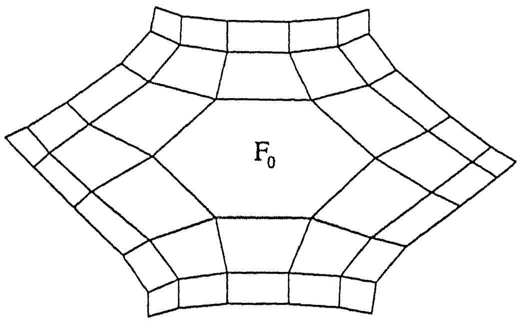


Figure 5.6: Initial control polyhedron $\mathcal{F}^0$

To study the nature of the limit surface near an extraordinary point it is enough to take $\mathcal{F}^0$ as a typical union of an n-sided face F_0 and two layers of quadrilaterals U_0 surrounding F^0 (Figure 5.6).

5.4 Parameterization

In this section we outline a procedure for the parametrization of the tensor product spline surface surrounding an extraordinary point.

Let us consider a polyhedron $\mathcal{F}^0$ as described above. First of all, we index the vertices of $\mathcal{F}^0$ as

$$B^j_{0,i}, \; i = 0, 1, \ldots, n-1, \; j = 1, 2, \ldots, 9$$

(see Figure 5.7). The indexes 0, i and j corresponds to the level, the block and jth entry of the block respectively. These vertices also include the vertices of U_0 having a rectangular topology. Therefore, a tensor product surface is obtained as a limit of $S_1^{(k)} U_0$ and is denoted by X. The parametric representation of X follows the normal tensor product procedure, the details of which is given below.

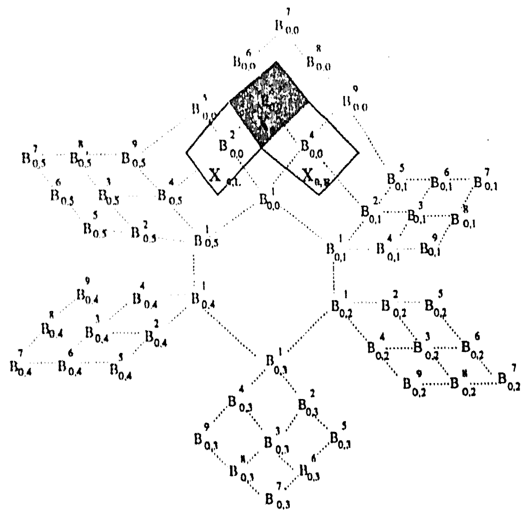


Figure 5.8: Patches X_0 (dark shaded region), $X_{0,L}$ and $X_{0,R}$ (light shaded regions)

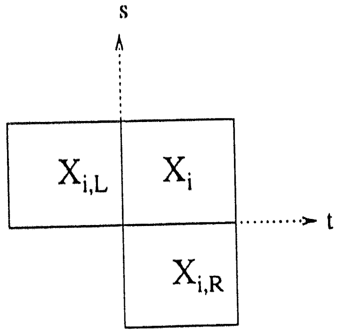


Figure 5.9: Parametrization of the patches X_i, $X_{i,L}$ and $X_{i,R}$

Let $J = 3n$ and $\mathcal{J} = \{1, 2, \ldots, J\}$.

Parameter space: The parameter space $\Omega_0 = w(0) \times \mathcal{J}$, of X consists of J copies of the square $w(0)$ provided with the neighborhood relations as given in (5.14). Denote by $\Omega_m := w(m) \times \mathcal{J}$, J copies of the square $w(m) = [0, \frac{h}{2^m}] \times [0, \frac{h}{2^m}]$, provided with the same neighbourhood relations as (5.14) except that h is replaced by $\frac{h}{2^m}$.

Representation: A representation of Ω_m is a map

$$\phi : (w, j) \to \phi^j(w) \in \mathbb{R}^2, \; j \in \mathcal{J}, \; w \in w(m)$$

from Ω_m to $\mathbb{R}^2$, where the functions ϕ^j are smooth injective maps in $\mathbb{R}^2$ with the property that pieces of boundary curves identified by the respective neighborhood

relations have identical images.

Now, coming to the actual construction of ϕ^js, we consider a bivariate affine map f of the form

$$f(x, y) = (ax + b)(cy + d) \tag{5.15}$$

which maps a quadrilateral in $\mathbb{R}^2$ into a quadrilateral in $\mathbb{R}^2$. It is easy to see that there exist affine maps f_i, $f_{i,L}$, $f_{i,R}$, $i = 0, 1, \ldots, n-1$ of the form (5.15) satisfying the neighbourhood relations (see Figure 5.10)

$$\begin{aligned} f_i(h, t) &= f_{i,R}(0, t) \\ f_{i,L}(s, h) &= f_i(s, 0) \\ f_{i,L}(0, h) &= f_{i,R}(0, 0) \\ f_{i-1,R}(s, h) &= f_{i,L}(s, 0) \\ f_{n-1,R}(s, h) &= f_{0,L}(s, 0) \end{aligned} \tag{5.16}$$

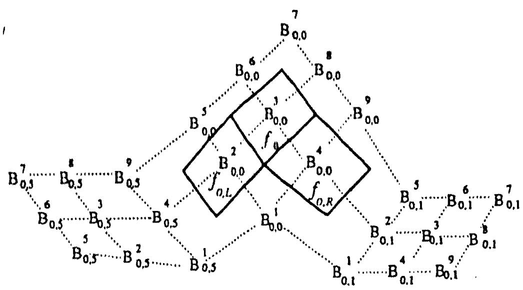


Figure 5.10: Representation f_0, $f_{0,L}$ and $f_{0.R}$

Clearly, the patches X_i, $X_{i,L}$, $X_{i,R}$ can now be parametrized by the above maps on $f_i(w(0))$, $f_{i,L}(w(0))$ and $f_{i,R}(w(0))$ respectively. The functions ϕ^i, $i \in \mathcal{J}$ are obtained by re-indexing the maps $f_{i,L}$, f_i and $f_{i,R}$, $i = 0, 1, \ldots, n-1$ sequentially. Thus, the map (see Figure 5.12)

$$\phi : (w, j) \rightarrow \phi^j(w), \quad j \in \mathcal{J},\ w \in w(0)$$

is a representation of the parameter space Ω_0. Therefore, X is a smooth function on a mesh Γ_0 (see Figure 5.12) defined by

$$\Gamma_0 = \cup_{j=1}^{n} \phi^j(w(0)). \tag{5.17}$$

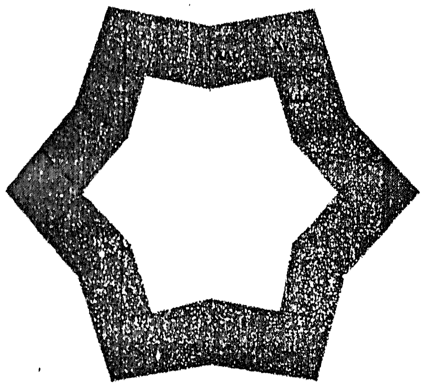


Figure 5.11: The surface X

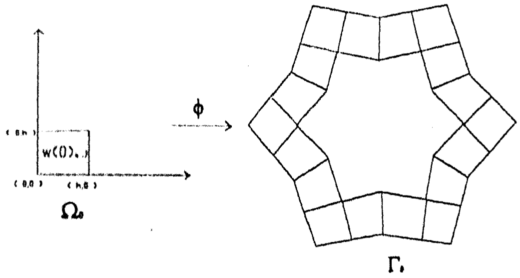


Figure 5.12: Parameter domain Γ_0

5.5 Prolongation

We have seen in the previous section that $S_1^{(k)} U^0$ tends as $k \to \infty$ to a surface X with a hole inside it. In this section we see by the method of prolongation that the surface $\lim_{k\to\infty} \mathcal{F}^k$ is a smooth surface without any hole and is obtained by prolonging the surface X.

Before considering the nature of the limit surface obtained by $\lim_{k\to\infty} \mathcal{F}^k$, let us consider the polyhedron $\mathcal{F}^1 = S_1[\mathcal{F}^0]$. It consists of an n-sided face F_1 and four layers of quadrilaterals (Figure 5.13). The polyhedron $\mathcal{F}^1 - F_1$ has a rectangular topology and contains $S_1[U^0]$ which is already known to have a rectangular topology. Therefore the corresponding limit surface (Figure 5.14)

$$X' = \lim_{k\to\infty} S_2^{(k)} [\mathcal{F}^1 - F_1]$$

$$\rightarrow\infty\, S_2^{(k)} [S_1[U^0]].$$

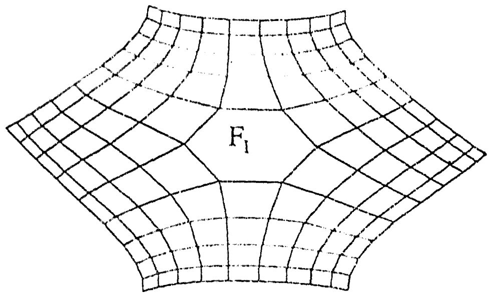


Figure 5.13: Control polyhedron $\mathcal{F}^1$

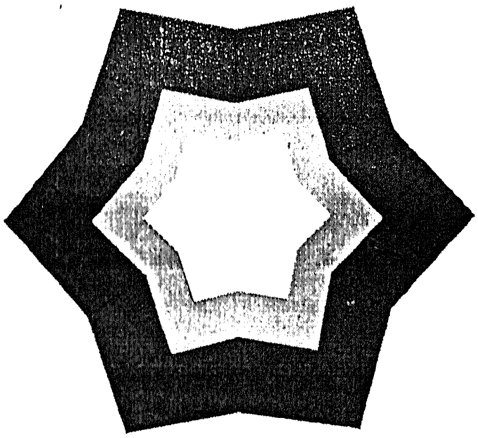


Figure 5.14: Limit surface X'

The surface $\lim_{k\to\infty} S_2^{(k)} [S_1[U^0]]$ is same as X, obtained by applying the subdivision scheme $\{S_k\}_{k\geq 2}$ to the subdivided polyhedron $S_1[U^0]$.

The scheme $\lim_{k\to\infty} S_2^{(k)}[S_1\, U^0]$ is now explained below in detail.

For the sake of simplicity we consider U^0 as a polyhedron having 9 control points in a tensor product topology. A single patch X is obtained as $\lim_{k\to\infty} S_1^{(k)} U^0$. The patch X has parameters defined on $w(0)$ (see Figure 5.15).

The polyhedron $S_1 U^0$ is again a polyhedron having 16 control points in tensor product topology. From $S_1 U^0$, we take 9 control points each (shown inside the dotted boxes in Figure 5.16) in tensor product topology and obtain the corresponding patches by the scheme $\lim_{k\to\infty} S_2^{(k)}$. These patches have parameters defined on $w(1)$.

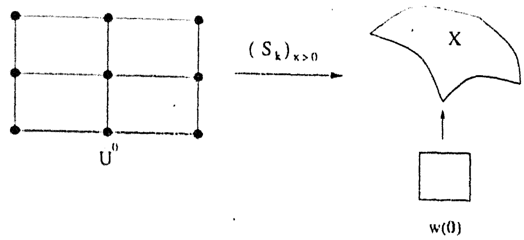


Figure 5.15: $S_1^{(k)} U^0$ as $k \to \infty$

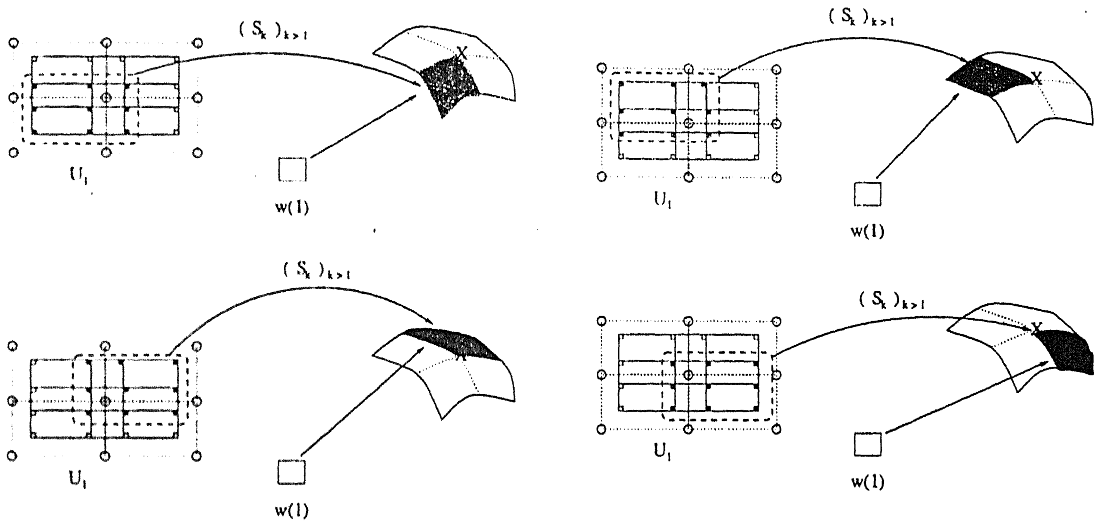


Figure 5.16: The scheme $S_2^{(k)}$ as $k \to \infty$

Because of the nature of tensor product schemes and subdivision schemes, $\lim_{k\to\infty} S_1^{(k)} U^0$ is a surface which is union of 4 patches with a neighbourhood relation. This surface is same as the original patch X but their parameters are now defined on a domain which is union of four $w(1)$ rectangles with the same neighbourhood relation as their corresponding patches (see Figure 5.17).

We now return to our original U^0. Following the above discussion the parameter space for the surface $\lim_{k\to\infty} S_2^{(k)} S_1 U^0$ which is denoted by Ω_1 is now obtained by dividing each unit of Ω_0 by four equal parts (see Figure 5.18). The neighbourhood relations are obvious for this parameter space.

Let ϕ_1 be a representation of Ω_1. Define the mesh Γ_1 (Figure 5.19) by

$$\phi_1 : \Omega_1 \to \Gamma_1. \tag{5.18}$$

Then using the parameterization procedure of Section 5.4, we see that the surface X

is also a smooth surface on Γ_1.

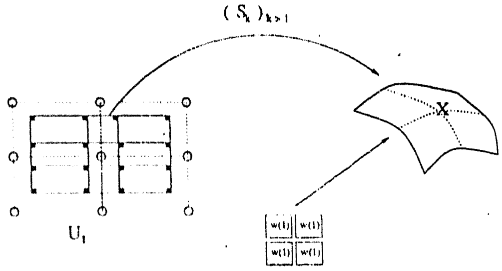


Figure 5.17: $S_2^{(k)} S_1 U^0$ as $k \to \infty$

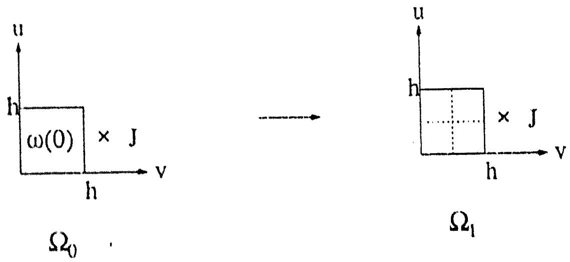


Figure 5.18: Subdivision of the parameter space

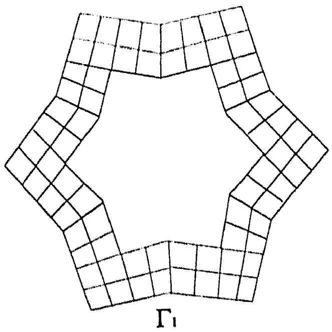


Figure 5.19: Parameter domain Γ_1

The mesh Γ_1 (Figure 5.20) is *regular* in the sense that each interior vertex of Γ_1 is shared by exactly four patches. We assume that the regular mesh Γ_1 can be prolonged by a layer $pr(\Gamma_0)$ such that $S(\Gamma_0) := \Gamma_1 \cup pr(\Gamma_0)$ is still regular (see Figure 5.20).

Using the parameterization procedure of Section 5.4 we see that the surface X' is a smooth function on $S(\Gamma_0)$ and the surface $pr^1(X) := X' - X$ which is a smooth extension of X called a prolongation of X to $pr(\Gamma_0)$ (light shaded surface in Figure 5.14).

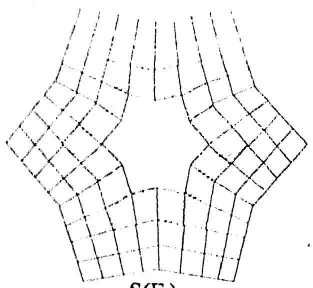


Figure 5.20: Prolngation of Γ_0

One can continue the prolongation procedure iteratively. In general $pr^k(X) = pr^1(pr^{k-1}(X))$, $k \geq 2$. Let P denote

$$P := \bigcup_{m \in \mathbb{N}} pr^m(X). \tag{5.19}$$

Below we investigate the nature of this surface P.

Definition 5.1. [63] A subdivision procedure $\{S_k\}$ is said to be convergent, if there is a p such that for any sequence $\{\mathrm{x}_m\}$, $\mathrm{x}_m \in pr^m(X)$ we have $\lim_{m \to} \mathrm{x}_m = p$. Therefore, if a subdivision scheme converges, then $\bar{P} = P \cup \{p\}$ is a surface without gap.

The point p described above is usually called an extraordinary point and the definition of convergence stated above ensures that the surface $\bar{P}$ is necessarily continuous at p. The convergence of a subdivision scheme depends upon the nature of the subdivision matrices which we study in the next section.

5.6 Subdivision Matrices

5.6.1 Matrix Formulation

We transform the subdivision process into a sequence of matrix transformations, whose underlying matrices are called subdivision matrices. First of all note that each of the tensor product surfaces $pr^m(X)$, $m \geq 1$ is defined by $9n =: K$ control points. We denote the set of control points of $pr^m(X)$ by a $9n \times 3$ block matrix B_m consisting of n blocks. Here m represents mth level.

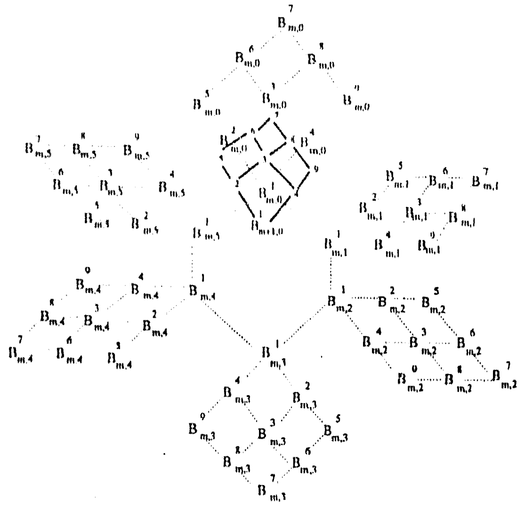


Figure 5.21: Control points at m th and $(m+1)$ th level of subdivision

Using the indexing procedure used for U^0, B_m is partitioned into n blocks

$$B_{m,0}, B_{m,1}, \ldots, B_{m,n-1}$$

each containing 9 control points (See Figure 5.21) i.e.,

$$B_m := [B_{m,0},\ B_{m,1}, \ldots, B_{m,n-1}]^T \tag{5.20}$$

where

$$B_{m,j} := [B^1_{m,j},\ B^2_{m,j}, \ldots, B^9_{m,j}]^T \tag{5.21}$$

and

$$B^i_{m,j} := (B^{i,1}_{m,j},\ B^{i,2}_{m,j},\ B^{i,3}_{m,j}) \in \mathbb{R}^3. \tag{5.22}$$

The superscript T used in (5.20) and (5.21) denotes block transpose. The matrices B_{m+1} and B_m are related to each other by the following equation,

$$B_{m+1} = M_{m+1} B_m \tag{5.23}$$

where M_{m+1} is a $K \times K$ matrix termed as $(m+1)$th level subdivision matrix. Considering the block form of B_m, i.e. $B_m = [B_{m,0}, \cdots, B_{m,n-1}]^T$, the matrix M_{m+1} can also

be written as a block matrix

$$M_{m+1} = \begin{pmatrix} (M_{m+1})_{1,0} & \cdots & (M_{m+1})_{1,n-1} \\ \vdots & \vdots & \vdots \\ (M_{m+1})_{n,0} & \vdots & (M_{m+1})_{n,n-1} \end{pmatrix}$$

where $(M_{m+1})_{i,j}$ denotes the (i,j)-th block of M_{m+1} and each block $(M_{m+1})_{i,j}$ contains 9×9 scalar elements. In particular we have

$$\begin{aligned} B_{m+1,0} &= (M_{m+1})_{1,0}B_{m,0} + \cdots + (M_{m+1})_{1,n-1}B_{m,n-1} \\ B_{m+1,1} &= (M_{m+1})_{2,0}B_{m,0} + \cdots + (M_{m+1})_{2,n-1}B_{m,n-1} \\ &= (M_{m+1})_{1,n-1}B_{m,0} + (M_{m+1})_{1,0}B_{m,1} + \cdots + (M_{m+1})_{1,n-2}B_{m,n-1} \end{aligned}$$

which follows from the nature of the subdivision scheme and the indexing scheme adopted for the control points. Therefore we have

$$\begin{aligned} (M_{m+1})_{2,1} &= (M_{m+1})_{1,0} \\ (M_{m+1})_{2,2} &= (M_{m+1})_{1,1} \\ &\vdots \\ (M_{m+1})_{2,n-1} &= (M_{m+1})_{1,n-2} \\ (M_{m+1})_{2,0} &= (M_{m+1})_{1,n-1}. \end{aligned}$$

One can see similar relation holding for consecutive rows of blocks. Therefore denoting $(M_{m+1})_{1,j}$ by M^j_{m+1}, $j = 0, 1, \ldots, n-1$, M_{m+1} acquires the block circulant form [25]

$$M_{m+1} := \begin{pmatrix} M^0_{m+1} & M^1_{m+1} & \cdots & M^{n-1}_{m+1} \\ M^{n-1}_{m+1} & M^0_{m+1} & \cdots & M^{n-2}_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ M^1_{m+1} & M^2_{m+1} & \cdots & M^0_{m+1} \end{pmatrix}.$$

Following [25] we denote M_{m+1} by $\text{bcirc}(M^0_{m+1}, \cdots, M^{n-1}_{m+1})$. Therefore to find out the matrix M_{m+1} it is enough to determine the block matrices $M^0_{m+1}, \cdots, M^{n-1}_{m+1}$ only. Note that these block matrices satisfy the equation

$$B_{m+1,0} = M^0_{m+1}B_{m,0} + M^1_{m+1}B_{m,1} + \cdots + M^{n-1}_{m+1}B_{m,n-1}. \tag{5.24}$$

It is easy to observe (see Figure 5.21) that

$$
\begin{aligned}
B^1_{m+1,0} &= b^0 B^1_{m,0} + b^1 B^1_{m,n-1} + b^1 B^1_{m,1} + b^2 B^1_{m,2} + \cdots + b^2 B^1_{m,n-2} \\
B^2_{m+1,0} &= rB^1_{m,0} + sB^2_{m,0} + sB^1_{m,n-1} + tB^4_{m,n-1} \\
B^3_{m+1,0} &= rB^1_{m,0} + sB^2_{m,0} + sB^4_{m,0} + tB^3_{m,0} \\
B^4_{m+1,0} &= rB^1_{m,0} + sB^4_{m,0} + sB^1_{m,1} + tB^2_{m,1} \\
B^5_{m+1,0} &= rB^2_{m,0} + sB^1_{m,0} + sB^4_{m,n-1} + tB^1_{m,n-1} \\
B^6_{m+1,0} &= rB^2_{m,0} + sB^1_{m,0} + sB^3_{m,0} + tB^4_{m,0} \\
B^7_{m+1,0} &= rB^3_{m,0} + sB^2_{m,0} + sB^4_{m,0} + tB^1_{m,0} \\
B^8_{m+1,0} &= rB^4_{m,0} + sB^3_{m,0} + sB^1_{m,0} + tB^2_{m,0} \\
B^9_{m+1,0} &= rB^4_{m,0} + sB^1_{m,0} + sB^2_{m,1} + tB^1_{m,1}
\end{aligned}
$$

where $b^0 = \alpha(n, m+1)$, $b^1 = \beta(n, m+1)$, $b^2 = \gamma(n, m+1)$, $r = \alpha(4, m+1)$, $s = \beta(4, m+1)$ and $t = \gamma(4, m+1)$.

Comparing the above system of equations with (5.24) we get.

$$
M^0_{m+1} = \left(\begin{array}{cccc|c}
b^0 & 0 & 0 & 0 & \\
r & s & 0 & 0 & \\
r & s & t & s & \\
r & 0 & 0 & s & \\
s & r & 0 & 0 & 0 \\
s & r & s & t & \\
t & s & r & s & \\
s & t & s & r & \\
s & 0 & 0 & r &
\end{array}\right), \quad
M^1_{m+1} = \left(\begin{array}{cccc|c}
b^1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
s & t & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
t & s & 0 & 0 &
\end{array}\right),
$$

$$
M^{n-1}_{m+1} = \left(\begin{array}{cccc|c}
b^1 & 0 & 0 & 0 & \\
s & 0 & 0 & t & \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
t & 0 & 0 & s & 0 \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 &
\end{array}\right), \quad \text{and} \quad
M^j_{m+1} = \left(\begin{array}{c|c}
b^2 & \\
\hline
 & 0 \\
0 &
\end{array}\right), \; j = 2, 3, \ldots, n-2.
$$

Let $M_{m+1}(i,j)$ represents the (i,j)-th entry of the matrix M_{m+1}. It is easily seen that all the entries of M_{m+1} are non-negative and the following relations are satisfied:

$$\begin{aligned}
\sum_{j=1}^{K} M_{m+1}(1,j) &= b^0 + 2b^1 + (n-3)b^2 = \frac{\cos^2(h/2^{m+1})}{\cos^2(h/2^m)} \\
\sum_{j=1}^{K} M_{m+1}(i,j) &= r + 2s + t = \frac{\cos^2(h/2^{m+1})}{\cos^2(h/2^m)}, \quad i \neq 1.
\end{aligned} \tag{5.25}$$

Now the patch corresponding to the j th unit of Ω_m is written in vector form as

$$\mathbf{x}_m^j := \mathbf{x}_m^j(u,v) := b_m(u,v,j)\, B_m, \quad (u,v) \in w(m), \tag{5.26}$$

where $b_m(u,v,j)$ is a row vector of tensor product trigonometric B-spline functions $b_m^k(u,v,j)$, $k = 1,2,\ldots K$:

$$b_m(u,v,j) := [b_m^1(u,v,j), \ldots, b_m^K(u,v,j)].$$

The basis functions $b_m^k(u,v,j)$, $k = 1,2,\ldots,K$ are localized and only 9 of them are nonzero in any unit of Ω_m. Denoting by $M^{(m+1)} = M_{m+1}M_m \cdots M_1$ and repeating (5.23) we get,

$$B_{m+1} = M_{m+1}M_m \cdots M_1 B_0 =: M^{(m+1)}\, B_0. \tag{5.27}$$

Therefore, convergence of the subdivision scheme depends upon the eigen values of the matrices $M^{(m+1)}$ and hence on its factors M_{m+1}. We study this aspect in the following subsection.

5.6.2 Eigenvalue Analysis

Recall that $M_{m+1} = bcirc(M_{m+1}^0, \cdots, M_{m+1}^{n-1})$. We now use the Fourier matrix method [25] to diagonalize the matrix M_{m+1}. For this let us define $w := exp(i\,2\pi/n)$ and

$$\hat{M}_{m+1}^k = \sum_{j=0}^{n-1} w^{-jk} M_{m+1}^j, \quad k = 0,1,\ldots,n-1. \tag{5.28}$$

Moreover, for any discrete set $\{p_0, \cdots, p_{n-1}\}$ define $\hat{p}^k$ by

$$\hat{p}^k = \sum_{j=0}^{n-1} w^{-jk} p_j. \tag{5.29}$$

Note that if $\hat{v}_{j,k}$, $k = 1, 2, \ldots 9$ are eigenvectors of $\hat{M}^j_{m+1}$, $j = 0, 1, \ldots, [n/2]$, then following Corollary 1.5 the vectors $Re(v_{j,k})$, $Im(v_{j,k})$, $j = 0, \ldots, [n/2]$, $k = 1, 2, \ldots, 9$ where

$$v_{j,k} := [\hat{v}_{j,k},\ w^j\, \hat{v}_{j,k}, \ldots, w^{(n-1)j}\, \hat{v}_{j,k}]^T \tag{5.30}$$

form a complete set of eigenvectors of M_{m+1}.

Let us define $\hat{b}^k := b^0 + b^1 w^{-k} + b^1 w^{-(n-1)k} + b^2 \sum_{j=2}^{n-2} w^{-jk}$, $k = 0, 1, \ldots, n-1$. Then for $k = 0, 1, \ldots, n-1$ we have

$$\hat{M}^k_{m+1} = \left(\begin{array}{cccc|c} \hat{b}^k & 0 & 0 & 0 & \\ r + s\,w^k & s & 0 & t\,w^k & \\ 0 & s & t & s & \\ r + s\,w^{-k} & t\,w^{-k} & 0 & s & \\ s + t\,w^k & r & 0 & s\,w^k & 0 \\ s & r & s & t & \\ t & s & r & s & \\ s & t & s & r & \\ s + t\,w^{-k} & s\,w^{-k} & 0 & r & \end{array}\right). \tag{5.31}$$

The eigen values of $\hat{M}^k_{m+1}$ are

$$\hat{b}^k,\ \frac{1}{8\,\cos^2(\frac{h}{2^{m+1}})\cos(\frac{h}{2^m})},\ \frac{1}{4\,\cos^2(\frac{h}{2^m})},\ \frac{1}{16\,\cos^2(\frac{h}{2^{m+1}})\,\cos^2(\frac{h}{2^m})}, 0, 0, 0, 0, 0$$

Therefore by Corollary 1.5 the numbers

$$\hat{b}^j,\ j = 0, 1, \ldots, [n/2]$$

and

$$\frac{1}{4\,\cos^2(\frac{h}{2^m})},\ \frac{1}{8\,\cos^2(\frac{h}{2^{m+1}})\cos(\frac{h}{2^m})},\ \frac{1}{16\,\cos^2(\frac{h}{2^{m+1}})\,\cos^2(\frac{h}{2^m})}$$

are eigenvalues of M_{m+1}. All other eigenvalues of M_{m+1} are 0. A simple calculation shows that

$$\hat{b}^k = b^0 + 2\cos(\frac{2k\pi}{n})(b^1 - b^2) - b^2,\ k = 0, 1, \ldots, n-1. \tag{5.32}$$

Therefore, the largest eigenvalue of M_{m+1} is $\lambda^1_{m+1} := \hat{b}^0 = \frac{\cos^2(h/2^{m+1})}{\cos^2(h/2^m)}$ and the double eigenvalue

$$\hat{b}^1 = \frac{2 + \cos(2\pi/n)}{8\cos^2(h/2^{m+1})\cos(h/2^m)} + \frac{1}{4\,\cos^2(h/2^{m+1})} \tag{5.33}$$

is the second largest eigenvalue of M_{m+1} for all n provided $0 \leq h/2^{m+1} < \pi/7$. This range has been found computationally.

Let $M = \lim_{m \to \infty} M_{m+1}$. The limit exists because the sequence of entries in M_{m+1} converges. Note that M also has a complete set of eigenvectors $\{e, v_2, v_3, \ldots, v_k\}$. The eigenvalues $1, \lambda_2, \lambda_3, \ldots, \lambda_K$ of M are the limit of the eigenvalues of M_{m+1}. In particular, the largest eigenvalue of M is 1 which is associated with the eigen vector $e = (1, 1, \ldots, 1)$. Let v_2 and v_3 be the eigenvectors associated with the second largest eigenvalue $\lambda_2 = \lambda_3$ of M.

Below we find λ_2, v_2 and v_3 for the cases $n = 3, 5$ and $n = 6$. The eigenvectors are obtained using Corollary 1.5.

n=3:

In this case $\lambda_2 = \lambda_3 = 7/16$. The corresponding eigenvectors v_2 and v_3 of M (up to three decimal places) are

$$\begin{aligned} v_2 = \big(&0.107,\ 0.065,\ 0.527,\ 0.494,\ -0.032,\ 0.497,\ 1.088,\ 1.069,\ 1.0, \\ &-0.149,\ -0.463,\ -0.735,\ -0.317,\ -0.849,\ -1.189,\ -1.518,\ -0.995,\ -0.5, \\ &0.042,\ 0.397,\ 0.208,\ -0.177,\ 0.882,\ 0.692,\ 0.43,\ -0.074,\ -0.5\,\big)^t. \end{aligned}$$

and

$$\begin{aligned} v_3 = \big(&0.110,\ 0.497,\ 0.545,\ 0.081,\ 0.999,\ 1.086,\ 1.124,\ 0.532,\ 0.0, \\ &0.037,\ -0.191,\ 0.184,\ 0.387,\ -0.528,\ -0.112,\ 0.380,\ 0.66,\ 0.866,\ -0.147, \\ &-0.305,\ -0.729,\ -0.468,\ -0.472,\ -0.973,\ -1.505,\ -1.192,\ -0.866\,\big)^t \end{aligned}$$

n=5:

In this case $\lambda_2 = \lambda_3 = 0.5385$. The corresponding eigenvectors v_2 and v_3 of M (up to three decimal places) are

$$\begin{aligned} v_2 = \big(&0.319,\ 0.458,\ 0.764,\ 0.683,\ 0.592,\ 0.872,\ 1.15,\ 1.067,\ 1.0, \\ &-0.055,\ -0.344,\ -0.131,\ 0.147,\ -0.584,\ -0.380,\ -0.199,\ 0.046,\ 0.309, \\ &-0.353,\ -0.671,\ -0.846,\ -0.592,\ -0.953,\ -1.107,\ -1.272,\ -1.039,\ -0.809, \\ &-0.163,\ -0.070,\ -0.391,\ -0.513,\ -0.005,\ -0.304,\ -0.588,\ -0.687,\ -0.809, \\ &0.252,\ 0.627,\ 0.604,\ 0.274,\ 0.95,\ 0.919,\ 0.909,\ 0.614,\ 0.309\,\big)^t. \end{aligned}$$

and

$$v_3 = \big(0.162,\ 0.511,\ 0.387,\ 0.067,\ 0.806,\ 0.683,\ 0.582,\ 0.299,\ 0.0,$$
$$0.354,\ 0.593,\ 0.846,\ 0.67,\ 0.812,\ 1.040,\ 1.273,\ 1.107,\ 0.951,$$
$$0.057,\ -0.144,\ 0.136,\ 0.347,\ -0.304,\ -0.040,\ 0.205,\ 0.385,\ 0.588,$$
$$-0.318,\ -0.682,\ -0.762,\ -0.455,\ -0.999,\ -1.065,\ -1.147,\ -0.869,\ -0.588,$$
$$-0.254,\ -0.278,\ -0.607,\ -0.629,\ -0.314,\ -0.618,\ -0.913,\ -0.922,\ -0.951\big)^t.$$

n=6:

In this case $\lambda_2 = \lambda_3 = 5/8$. The eigenvectors v_2 and v_3 of M are

$$v_2 = \big(0.378,\ 0.554,\ 0.803,\ 0.722,\ 0.710,\ 0.925,\ 1.143,\ 1.06,\ 1.0,$$
$$0.054,\ -0.128,\ 0.115,\ 0.311,\ -0.254,\ -0.033,\ 0.164,\ 0.318,\ 0.5,$$
$$-0.324,\ -0.681,\ -0.688,\ -0.411,\ -0.965,\ -0.958,\ -0.98,\ -0.742,\ -0.5,$$
$$-0.378,\ -0.554,\ -0.803,\ -0.722,\ -0.710,\ -0.925,\ -1.143,\ -1.06,\ -1.0,$$
$$-0.054,\ 0.128,\ -0.115,\ -0.311,\ 0.254,\ 0.033,\ -0.164,\ -0.318,\ -0.5$$
$$0.324,\ 0.681,\ 0.688,\ 0.411,\ 0.965,\ 0.95,\ 0.979,\ 0.742,\ 0.5\big)^t$$

and

$$v_3 = \big(0.156,\ 0.467,\ 0.331,\ 0.058,\ 0.704,\ 0.572,\ 0.470,\ 0.245,\ 0.0,$$
$$0.405,\ 0.713,\ 0.861,\ 0.654,\ 0.967,\ 1.087,\ 1.225,\ 1.04,\ 0.866,$$
$$0.25,\ 0.246,\ 0.530,\ 0.596,\ 0.263,\ 0.515,\ 0.755,\ 0.795,\ 0.866,$$
$$-0.156,\ -0.467,\ -0.330,\ -0.058,\ -0.704,\ -0.572,\ -0.470,\ -0.245,\ 0.0,$$
$$-0.405,\ -0.713,\ -0.861,\ -0.654,\ -0.967,\ -1.087,\ -1.225,\ -1.04,\ -0.866,$$
$$-0.25,\ -0.246,\ -0.530,\ -0.596,\ -0.263,\ -0.515,\ -0.755,\ -0.795,\ -0.866\big)^t.$$

5.7 Convergence Analysis

Let us define the space $\mathbb{R}^{K\times 3}$ by

$$\mathbb{R}^{K\times 3} := \{\mathrm{x} := (x_1, \dots, x_K)^T,\ x_i := (x_i^1,\ x_i^2,\ x_i^3) \in \mathbb{R}^3\}.$$

The elements of $\mathbb{R}^{K\times 3}$ is considered as a matrix of order $K \times 3$.

Let e_j be the **column vector** in $\mathbb{R}^K$ whose all coordinates are zero except the jth coordinate. It is well known that the set of vectors e_j, $j = 1, 2, \ldots, K$ forms a basis for $\mathbb{R}^K$. Let $\mathbf{x} = (x_1, \ldots, x_K)^T \in \mathbb{R}^{K\times 3}$. Then $\mathbf{x}$ can be expressed as

$$\mathbf{x} = e_1 x_1 + \ldots + e_K x_K.$$

Define $\|\mathbf{x}\|_e = \max_{1\leq i\leq K} \|x_i\|$ where $\|x_i\| = \max\{|x_i^1|, |x_i^2|, |x_i^3|\}$.

We have mentioned in the previous section that M has a complete set of eigenvectors $e, v_2, \ldots, v_K$. These eigen vectors form a basis for $\mathbb{R}^K$. Let us denote this basis by

$$\mathcal{B} := \{v_1,\ v_2, \ldots, v_K\}$$

where $v_1 = e$. Then for $\mathbf{x} \in \mathbb{R}^{K\times 3}$ there exist **row vectors** $p_i \in \mathbb{R}^3$, $i = 1, 2, \ldots, K$ such that

$$\mathbf{x} = v_1 p_1 + \ldots + v_K p_K.$$

Define the norm $\|\mathbf{x}\|_v := \max_{1\leq i\leq K} \|p_i\|$ where $\|p_i\| = \max\{|p_i^1|, |p_i^2|, |p_i^3|\}$.

Now consider the two sets of basis elements of $\mathbb{R}^{K\times 3}$, $\{e_i\}_1^K$ and $\{v_i\}_1^K$. Let $\mathbf{T}$ be the transformation matrix corresponding to this change of basis i.e,

$$(\beta_1, \ldots, \beta_K)^T = \mathbf{T}(\alpha_1, \ldots, \alpha_K)^T$$

where $\sum_{i=1}^K e_i\alpha_i = \sum_{i=1}^K v_i\beta_i$. Clearly $\mathbf{T}$ is an invertible $K \times K$ matrix and

$$\|\mathbf{x}\|_v \leq \|\mathbf{T}\|\ \|\mathbf{x}\|_e \quad \text{and} \quad \|\mathbf{x}\|_e \leq \|\mathbf{T}^{-1}\|\ \|\mathbf{x}\|_v. \tag{5.34}$$

A sequence $\mathbf{r}_1, \mathbf{r}_2, \cdots$, in $\mathbb{R}^{K\times 3}$ is said to be order $o(l^m)$ if

$$\lim_{m\to\infty} \frac{\|\mathbf{r}_m\|_e}{l^m} = 0.$$

The above sequence is said to be order $O(l^m)$ if

$$\lim_{m\to\infty} \frac{\|\mathbf{r}_m\|_e}{l^m} < \infty.$$

Let $\mathbf{x} = \mathcal{B}_0$. We show that the sequence of configurations $M^{(m)}\mathbf{x}$ converges to a limiting configuration. First we break the operator $M^{(m)}$ into three parts, one stationary and two nonstationary.

Let us define $S_m = M_m - M$, $m \in \mathbb{Z}_+$. Since $\alpha(n,m) > \alpha$, $\beta(n,m) > \beta$ and $\gamma(n,m) > \gamma$, all the entries of M_m is greater than or equal to the corresponding entries in M. Therefore all the entries in S_m are non-negative. The following recursive relation for $M^{(m)}$ holds.

Lemma 5.1. *For $m \geq 2$*

$$M^{(m)} = M^m + \sum_{j=1}^{m-1} M^{m-j}\, S_j\, M^{(j-1)} + S_m\, M^{(m-1)}. \tag{5.35}$$

Proof: We prove the lemma by induction on m.

Note that $M^{(1)} = M_1 = M + S_1$. Thus

$$M^{(2)} = M_2\, M^{(1)} = (M + S_2)(M + S_1) = M^2 + M\, S_1 + S_2\, M_1.$$

This proves (5.35) for $m = 2$.

Assume that (5.35) holds for m. Now

$$\begin{aligned}
M^{(m+1)} &= M_{m+1}\, M^{(m)} = (M + S_{m+1})M^{(m)} \\
&= M\{M^m + \sum_{j=1}^{m-1} M^{m-j}\, S_j\, M^{(j-1)} + S_m M^{(m-1)}\} + S_{m+1}\, M^{(m)} \\
&= M^{m+1} + \sum_{j=1}^{m-1} M^{m+1-j}\, S_j\, M^{(j-1)} + M\, S_m\, M^{(m-1)} + S_{m+1}\, M^{(m)}
\end{aligned}$$

Therefore the lemma holds for $m + 1$. This proves the lemma. □

Let us denote $\mathbf{y}_m := \sum_{j=1}^{m-1} M^{m-j}\, S_j\, M^{(j-1)}\mathbf{x}$.

To show the convergence of the sequence $\{M^{(m)}\mathbf{x}\}$ to a limiting configuration in $\mathbb{R}^{K\times 3}$ we show the convergence of the sequences $\{M^m\mathbf{x}\}$, $\{\mathbf{y}_m\}$ and $\{S_m\, M^{(m-1)}\mathbf{x}\}$. It has been shown by Reif [63] that

$$M^m\mathbf{x} = ep_1 + \lambda_2^m(v_2 p_2 + v_3 p_3) + o(\lambda_2^m) = e\,p_1 + o(1) \tag{5.36}$$

where λ_2 is the second largest eigenvalue and v_2 and v_3 are the corresponding eigenvectors of M. Therefore $\{M^m\mathbf{x}\}$ converges.

To show the convergence of $\{\mathbf{y}_m\}$ and $\{S_m M^{(m-1)}\mathbf{x}\}$ we need the following estimates, which we derive in the following lemmas.

Lemma 5.2. *For any* $\mathbf{x} \in \mathbb{R}^{K\times 3}$ *and* $m \geq 1$, *the following holds:*

$$\left\|M^{(m)}\mathbf{x}\right\|_e \leq \frac{\cos^2(h/2^m)}{\cos^2(h)} \left\|\mathbf{x}\right\|_e < \frac{1}{\cos^2(h)} \left\|\mathbf{x}\right\|_e. \tag{5.37}$$

Proof: Recall that

$$M^{(m)}\mathbf{x} = M_m M_{m-1} \dots M_1\mathbf{x}.$$

Let us denote the (i,j) th element of the matrices $M_r\mathbf{x}$ and M_r by $(M_r\mathbf{x})_{i,j}$ and $(M_r)_{i,j}$ respectively. Then,

$$|(M_r\mathbf{x})_{i,j}| \leq |\sum_{s=1}^{K}(M_r)_{i,s}\, x_s^j| \leq \sum_{s=1}^{K}|(M_r)_{i,s}|\,|x_s^j|.$$

Since $|x_s^j| \leq \left\|\mathbf{x}\right\|_e$ we have

$$|(M_r\mathbf{x})_{i,j}| \leq \sum_{s=1}^{K}|(M_r)_{i,s}|\,\left\|\mathbf{x}\right\|_e.$$

But

$$\sum_{s=1}^{K}|(M_r)_{i,s}| = \sum_{s=1}^{K}(M_r)_{i,s} = \frac{\cos^2(h/2^r)}{\cos^2(h/2^{r-1})} \quad \forall i = 1,2,\dots,K.$$

So $|(M_r\mathbf{x})_{i,j}| \leq \frac{\cos^2(h/2^r)}{\cos^2(h/2^{r-1})}\left\|\mathbf{x}\right\|_e$, $i = 1,2,3;\ i = 1,2,\dots,K$. Therefore,

$$\left\|M_r\mathbf{x}\right\|_e \leq \frac{\cos^2(h/2^r)}{\cos^2(h/2^{r-1})}\left\|\mathbf{x}\right\|_e, \quad r \in \mathbb{Z}_+. \tag{5.38}$$

Hence (5.37) follows. □

Taking limit in (5.38) as $r \to \infty$ we get the following result.

Lemma 5.3. *For any* $\mathbf{x} \in \mathbb{R}^{K\times 3}$ *and* $m \geq 1$ *we have*

$$\left\|M^m\mathbf{x}\right\|_e < \left\|\mathbf{x}\right\|_e. \tag{5.39}$$

Now we obtain an estimate for $\left\|S_m\mathbf{x}\right\|_e$ in the following lemma.

Lemma 5.4. *For $m \geq 1$ and any $\mathbf{x} \in \mathbb{R}^{K\times 3}$ we have*

$$\|S_m\mathbf{x}\|_e \leq \frac{C(h)}{4^m}\,\|\mathbf{x}\|_e \tag{5.40}$$

for some constant $C(h)$ independent of m. Hence the sequence $\{S_m\mathbf{x}\}$ converges to 0.

Proof: Let $(S_m\mathbf{x})_{i,j}$ denote the (i,j)th entry of S_m. Then for $i = 1, 2, \ldots, K$ and $j = 1, 2, 3$ we have

$$\begin{aligned}|(S_m\mathbf{x})_{i,j}| &\leq \sum_{s=1}^{K}\left|\Big((M_m)_{i,s} - (M)_{i,s}\Big)\,x_s^j\right| \leq \sum_{s=1}^{K}|(M_m)_{i,s} - (M)_{i,s}|\,|x_s^j| \\ &\leq \|\mathbf{x}\|_e \sum_{s=1}^{K}|(M_m)_{i,s} - (M)_{i,s}| = \|\mathbf{x}\|_e \sum_{s=1}^{K}\Big((M_m)_{i,s} - (M)_{i,s}\Big)\end{aligned}$$

since $(M_m)_{i,k} \geq (M)_{i,k}$ for all $i, k = 1, \ldots, K$. Further $\sum_{s=1}^{K}(M)_{i,s} = 1$ and

$$\sum_{s=1}^{K}(M_m)_{i,s} = \frac{\cos^2(h/2^m)}{\cos^2(h/2^{m-1})}.$$

Therefore,

$$|(S_m\mathbf{x})_{i,j}| \leq \|\mathbf{x}\|_e\left(\frac{\cos^2(\frac{h}{2^m})}{\cos^2(\frac{h}{2^{m-1}})} - 1\right) \leq \frac{\|\mathbf{x}\|_e}{4^m}\,\frac{3\,h^2}{\cos^2(h)} = \frac{C(h)}{4^m}\|\mathbf{x}\|_e$$

where $C(h) = \frac{3h^2}{\cos^2(h)}$. $\square$

In the following lemma we show that the the sequence $\{\mathbf{y}_m\}$ converges.

Lemma 5.5. *There exists unique $q \in \mathbb{R}^3$ such that*

$$\lim_{m\to\infty} \mathbf{y}_m = eq. \tag{5.41}$$

Besides, $\|\mathbf{y}_m - eq\|_e \leq \frac{C_1}{4^m}$ for some constant C_1.

Proof: First of all we show that the sequence $\{\mathbf{y}_m\}$ is Cauchy.

For $m, k \geq 1$ we have

$$
\begin{aligned}
\mathbf{y}_{m+k} - \mathbf{y}_m &= \sum_{j=1}^{m+k-1} M^{m+k-j}\, S_j\, M^{(j-1)}\, \mathbf{x} - \sum_{j=1}^{m-1} M^{m-j}\, S_j\, M^{(j-1)}\, \mathbf{x} \\
&= \sum_{j=1}^{m-1} \left(M^{m+k-j}\, S_j\, M^{(j-1)}\, \mathbf{x} - M^{m-j}\, S_j\, M^{(j-1)}\, \mathbf{x} \right) \\
&\quad + \sum_{j=m}^{m+k-1} M^{m+k-j}\, S_j\, M^{(j-1)}\, \mathbf{x}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\| \mathbf{y}_{m+k} - \mathbf{y}_m \right\|_e \leq & \sum_{j=1}^{m-1} \left\| M^{m+k-j}\, S_j\, M^{(j-1)}\, \mathbf{x} - M^{m-j}\, S_j\, M^{(j-1)}\, \mathbf{x} \right\|_e \\
& + \sum_{j=m}^{m+k-1} \left\| M^{m+k-j}\, S_j\, M^{(j-1)}\, \mathbf{x} \right\|_e .
\end{aligned} \tag{5.42}
$$

For any choice of j, denote $\mathbf{v} := S_j\, M^{(j-1)}\mathbf{x}$. So $\mathbf{v} = v_1\, \beta_1 + v_2\, \beta_2 + \cdots + v_K\, \beta_K$ for some $\beta_i \in \mathbb{R}^3$, $i = 1, \ldots, K$. Then by Lemma 5.2 and Lemma 5.4 we have

$$
\left\| \mathbf{v} \right\|_e \leq \frac{C(h)}{4^j \cos^2(h)} \left\| \mathbf{x} \right\|_e . \tag{5.43}
$$

Moreover

$$
M^{m+k-j}\, \mathbf{v} = v_1\, \beta_1 + \lambda_2^{m+k-j}\, v_2\, \beta_2 + \cdots + \lambda_K^{m+k-j}\, v_K\, \beta_K
$$

and $M^{m-j}\, \mathbf{v} = v_1\, \beta_1 + \lambda_2^{m-j}\, v_2\, \beta_2 + \cdots + \lambda_K^{m-j}\, v_K\, \beta_K$. Therefore

$$
\begin{aligned}
& \left\| M^{m+k-j}\, \mathbf{v} - M^{m-j}\, \mathbf{v} \right\|_e \\
&= \left\| (\lambda_2^{m+k-j} - \lambda_2^{m-j})\, v_2\, \beta_2 + \ldots + (\lambda_K^{m+k-j} - \lambda_K^{m-j})\, v_K\, \beta_K \right\|_e \\
&= \left\| \lambda_2^{m-j}\, (\lambda_2^k - 1)\, v_2\, \beta_2 + \ldots + \lambda_K^{m-j}\, (\lambda_K^k - 1)\, v_K\, \beta_K \right\|_e \\
&\leq \lambda_2^{m-j} \max_{2 \leq j \leq K} |\lambda_j^k - 1|\, (K-1) \max_{2 \leq j \leq K} \left\| v_j\, \beta_j \right\|_e \\
&\leq \lambda_2^{m-j} \max_{2 \leq j \leq K} |\lambda_j^k - 1|\, (K-1)\, \left\| T^{-1} \right\|\, \left\| T \right\|\, \left\| \mathbf{v} \right\|_e .
\end{aligned} \tag{5.44}
$$

Since $0 < \lambda_r < 1$ for $2 \leq r \leq K$ and K is fixed we get

$$
\left\| M^{m+k-j}\, \mathbf{v} - M^{m-j}\, \mathbf{v} \right\|_e \leq C_2\, \lambda_2^{m-j}\, \left\| \mathbf{v} \right\|_e .
$$

For some constant C_2. Thus by (5.43), we have

$$
\left\| M^{m+k-j}\, \mathbf{v} - M^{m-j}\, \mathbf{v} \right\|_e \leq \lambda_2^{m-j}\, \frac{C_2 C(h)}{4^j \cos^2(h)} \left\| \mathbf{x} \right\|_e . \tag{5.45}
$$

Further, using Lemmas 5.2, 5.3 and 5.4 we get

$$\left\| M^{m+k-j} S_j M^{(j-1)} \mathbf{x} \right\|_e \leq \frac{C(h)}{\cos^2(h)} \frac{\|\mathbf{x}\|_e}{4^j}. \tag{5.46}$$

Substituting (5.45) and (5.46) in (5.42) we get

$$\left\| \mathbf{y}_{m+k} - \mathbf{y}_m \right\|_e \leq C(h) \, \|\mathbf{x}\|_e \left(C_2 \sum_{j=1}^{m-1} \frac{\lambda_2^m}{(4\lambda_2)^j} + \sum_{j=m}^{m+k-1} \frac{1}{4^j} \right).$$

Since $4\lambda_2 > 2$ and both the sums $\sum_{j=1}^{\infty} \frac{1}{(4\lambda_2)^j}$ and $\sum_{j=0}^{\infty} \frac{1}{4^j}$ exist we get

$$\left\| \mathbf{y}_{m+k} - \mathbf{y}_m \right\|_e \leq C(h) \, \|\mathbf{x}\|_e \, (\lambda_2^m + \frac{1}{4^m}) \tag{5.47}$$

for a generic constant $C(h)$ independent of m. This proves that the sequence $\{\mathbf{y}_m\}_{m=2}^{\infty}$ is Cauchy and hence convergent.

Now, we show that (5.41) holds. Let us assume that $\lim_{k\to\infty} \mathbf{y}_m = \mathbf{y}$ and $\mathbf{y}$ is expressed in terms of eigenvectors of M as

$$\mathbf{y} = v_1 \, \beta_1' + v_2 \, \beta_2' + \ldots + v_K \, \beta_K'. \tag{5.48}$$

for some $\beta_i' \in \mathbb{R}^3$, $i = 1, \ldots, K$. Then we have

$$\begin{aligned} \left\| M \mathbf{y} - \mathbf{y} \right\|_e &\leq \left\| M \mathbf{y} - M \mathbf{y}_m \right\|_e + \left\| M \mathbf{y}_m - \mathbf{y}_{m+1} \right\|_e + \left\| \mathbf{y}_{m+1} - \mathbf{y} \right\|_e \\ &\leq \left\| \mathbf{y} - \mathbf{y}_m \right\|_e + \left\| M \, S_m \, M^{(m-1)} \mathbf{x} \right\|_e + \left\| \mathbf{y}_{m+1} - \mathbf{y} \right\|_e. \end{aligned} \tag{5.49}$$

Therefore by Lemmas 5.2 and 5.4 we get

$$\left\| M \mathbf{y} - \mathbf{y} \right\|_e \leq \left\| \mathbf{y} - \mathbf{y}_m \right\|_e + \frac{C(h)}{4^m \cos^2(h)} \|\mathbf{x}\|_e + \left\| \mathbf{y}_{m+1} - \mathbf{y} \right\|_e.$$

By taking $m \to \infty$ we have $M \mathbf{y} = \mathbf{y}$. Therefore $\mathbf{y}$ is an eigenvector of M for the eigenvalue 1. Thus $\mathbf{y} = v_1 \, \beta_1' := c \, \beta_1'$. Writing $\beta_1' = q$, we get the required result.

Since $\lambda_2 \leq \frac{1}{4}$, taking $k \to \infty$ in (5.47) we get $\left\| y - y_m \right\|_e \leq \frac{C(h)}{4^m}$. Hence second part of the lemma follows. ☐

By (5.36), (5.41) and the fact that the sequences $\{S_m M^{(m-1)} \mathbf{x}\}$ converges to 0 we get

$$B_m = c(p_1 + q) + \lambda_2^m (v_2 p_2 + v_3 p_3) + o(\lambda_2^m) + O(1)/4^m. \tag{5.50}$$

The main theorem concerning the convergence of the subdivision scheme follows as a consequence of the above considerations.

Theorem 5.6. *Let the sequence $\{\mathbf{x}_m\}_{m=1}^{\infty}$, $\mathbf{x}_m \in pr^m(B_0)$ be defined by (5.26). Then*

$$\lim_{m\to+\infty} \mathbf{x}_m = (p_1+q).$$

where p_1 and q are defined by (5.50).

Proof: Let $\mathbf{x}_m \in pr^m(B_0)$. Then by (5.50) we have

$$\mathbf{x}_m^j(u,v) = b_m(u,v,j)\,B_m = b_m(u,v,j)\Big\{c(p_1+q)+o(1)\Big\}. \tag{5.51}$$

Let $T_0(x;h)$ be the trigonometric B-spline of degree 2 with knots $\{0,h,2h,3h\}$ and let T_i denote the translation

$$T_i(x;h) = T_0(x-ih;h), \quad i\in\mathbb{Z}. \tag{5.52}$$

It can be easily checked (see [54]) that

$$\sum_{i\in\mathbb{Z}} T_i(x;h) = \frac{1}{\cos(h)}.$$

In view of the local support of $b_m^k(u,v,j)$ the sum $\sum_{k=1}^K b_m^k(u,v,j)$ has only 9 non-zero terms. These terms are supported on j th unit of Ω_m. Furher, in view of the above equation

$$b_m(u,v,j)c = \sum_{k=1}^{K} b_m^k(u,v,j) = \frac{1}{\cos^2(h/2^m)} < \frac{1}{\cos^2(h)}. \tag{5.53}$$

Hence $b_m(u,v,j)\Big\{c(p_1+q)+o(1)\Big\} = \frac{1}{\cos^2(h/2^m)}(p_1+q)+o(1)$. Therefore, $\mathbf{x}_m(u,v,j)\to(p_1+q)$. This completes the proof of the convergence. $\square$

The following important consequence is immediate.

$$\bar{P} = P\cup(p_1+q) = \bigcup_{m\in\mathbb{Z}} pr^m(X)\cup(p_1+q)$$

i.e. adding the limiting value to the surface fills the hole of the surface and hence the limit surface becomes continous.

Below, we study the smoothness of this surface and show that the surface $\bar{P}$ is tangent plane continous at $p := p_1+q$.

5.8 Tangent Plane Continuity

In case of stationary subdivision schemes continuity of the tangent plane of the limit surfaces is shown [63] with the help of the characteristic maps associated with the subdivision schemes. Since, the subdivision scheme described above is nonstationary, the notion of characteristic map is modified to suit the present scheme.

Definition 5.2. For the subdivision matrix M the sequences of maps $\Psi_m : \Omega_m \to \mathbb{R}^2$, $m \geq 1$ defined by

$$\Psi_m : (u, v, j) \to b_m(u, v, j)\, V := b_m(u, v, j)\, [v_2, v_3],$$

is called a sequence of characteristic maps, where v_2 and v_3 are eigenvectors of M corresponding to the eigenvalue $\lambda_2 = \lambda_3$ and V is a $K \times 2$ matrix with the vectors v_2 and v_3 as its columns.

The rows of V can be viewed as points in $\mathbb{R}^2$ and $\Psi_m(.,.,.)$ can be considered as a two dimensional tensor product trigonometric spline surface with rows of V as control points. Note that the vectors v_2 and v_3 for the particular cases $n = 3$, $n = 5$ and $n = 6$ are obtained in Section 5.6. The characteristic map Ψ_0 for these cases are shown in Figures 5.22-5.24. Since at all the points on the map the vectors $\frac{\partial}{\partial u}\mathbf{x}_m$ and $\frac{\partial}{\partial v}\mathbf{x}_m$ have different directions they are linear independent. Hence $\Delta_m(u, v, j) := det\, \frac{\partial \Psi_m(u,v,j)}{\partial(u,v)} \neq 0$ for all $(u, v, j) \in \Omega_m$.

Theorem 5.7. *The limit surface $\bar{P}$ has a continous tangent at p.*

Proof: By (5.50) we have

$$B_m = c\,(p_1 + q) + \lambda_2^m\,(v_2\,p_2 + v_3\,p_3) + o(\lambda_2^m) + \{o(\lambda_2^m) + \frac{O(1)}{4^m}\}.$$

Therefore

$$\begin{aligned}
\mathbf{x}_m(u, v, j) &= b_m(u, v, j)\Big\{c\,(p_1 + q) + \lambda_2^m\,(v_2\,p_2 + v_3\,p_3) + o(\lambda_2^m) + \{o(\lambda_2^m) + \frac{O(1)}{4^m}\}\Big\} \\
&= b_m(u, v, j) c\,(p_1 + q) + \lambda_2^m\, b_m(u, v, j)\,(v_2\,p_2 + v_3\,p_3) \\
&\quad + b_m(u, v, j)\Big\{o(\lambda_2^m) + \frac{O(1)}{4^m}\Big\}.
\end{aligned}$$

But we know by (5.53) $b_m(u,v,j)\,e = \frac{1}{\cos^2(h/2^m)}$. Therefore

$$\frac{\partial\, b_m(u,v,j)\,e}{\partial\, u} = 0 \quad \text{and} \quad \frac{\partial\, b_m(u,v,j)\,e}{\partial\, v} = 0.$$

Hence

$$\begin{aligned}
&\frac{\partial}{\partial\, u}\mathbf{x}_m(u,v,j)\\
&= \lambda_2^m \frac{\partial}{\partial\, u} b_m(u,v,j)(v_2 p_2 + v_3 p_3) + \frac{\partial}{\partial\, u} b_m(u,v,j)\Big\{o(\lambda_2^m) + \{o(\lambda_2^m) + \frac{O(1)}{4^m}\}\Big\},\\
&\frac{\partial}{\partial\, v}\mathbf{x}_m(u,v,j)\\
&= \lambda_2^m \frac{\partial}{\partial v} b_m(u,v,j)(v_2 p_2 + v_3 p_3) + \frac{\partial}{\partial\, v} b_m(u,v,j)\Big\{o(\lambda_2^m) + \{o(\lambda_2^m) + \frac{O(1)}{4^m}\}\Big\},
\end{aligned}$$

and the direction of the normal vector to the surface $\mathbf{x}_m$ for any value of (u,v) is given by the vector product

$$\begin{aligned}
\frac{\partial}{\partial u}\mathbf{x}_m \times \frac{\partial}{\partial v}\mathbf{x}_m = \lambda_2^{2m}\Big\{&\frac{\partial}{\partial u} b_m(u,v,j)(v_2p_2 + v_3p_3) \times \frac{\partial}{\partial v} b_m(u,v,j)(v_2p_2 + v_3p_3)\\
&+ \frac{\partial}{\partial u} b_m(u,v,j)\{o(\lambda_2^m) + \frac{O(1)}{4^m}\} \times \frac{\partial}{\partial v} b_m(u,v,j)\{o(\lambda_2^m) + \frac{O(1)}{4^m}\}\Big\}.
\end{aligned}$$

Since $\frac{\partial b_m'(u,v)}{\partial v}$ and $\frac{\partial b_m'(u,v)}{\partial u}$ are bounded functions on any unit of Ω_m and $4\lambda_2 > 1$ for any $j = 1, 2, \ldots, K$ we have

$$\begin{aligned}
&\frac{\partial}{\partial u}\mathbf{x}_m \times \frac{\partial}{\partial v}\mathbf{x}_m(u,v,j) = \mathbf{x}_{m,u} \times \mathbf{x}_{m,v}(u,v,j)\\
&= \lambda_2^{2m}\Big\{\frac{\partial}{\partial u} b_m(u,v,j) v_2 \frac{\partial}{\partial v} b_m(u,v,j) v_2 (p_2 \times p_2) + \frac{\partial}{\partial u} b_m(u,v,j) v_2 \frac{\partial}{\partial v} b_m(u,v,j) v_3 (p_2 \times p_3)\\
&\quad + \frac{\partial}{\partial u} b_m(u,v,j) v_3 \frac{\partial}{\partial v} b_m(u,v,j) v_2 (p_3 \times p_2) + \frac{\partial}{\partial u} b_m(u,v,j) v_3 \frac{\partial}{\partial v} b_m(u,v,j) v_3 (p_3 \times p_3)\Big\}\\
&\quad + \Big(o(\lambda_2^{2m}) + O(1)\frac{\lambda_2^m}{4^m} + \frac{O(1)}{4^{2m}}\Big)\\
&= \lambda_2^{2m}(p_2 \times p_3)\Big(\frac{\partial}{\partial u} b_m(u,v,j) v_2 \frac{\partial}{\partial v} b_m(u,v,j) v_3 - \frac{\partial}{\partial v} b_m(u,v,j) v_2 \frac{\partial}{\partial u} b_m(u,v,j) v_3\Big)\\
&\quad + \lambda_2^{2m}\Big(o(1) + \frac{O(1)}{(4\lambda_2)^m} + \frac{O(1)}{(4\lambda_2)^{2m}}\Big)\\
&= \lambda_2^{2m}\Big\{\Delta_m(u,v,j)(p_2 \times p_3) + o(1)\Big\}
\end{aligned}$$

Moreover Δ_m is nonzero and positive for all (u,v,j). Therefore the normalized normal vector is given by

$$\begin{aligned}
n_m(u,u,j) &= \frac{x_{m,u}(u,v,j) \times x_{m,v}(u,v,j)}{\|x_{m,u}(u,v,j) \times x_{m,v}(u,v,j)\|}\\
&= \frac{\Delta_m(u,v,j)(p_2 \times p_3) + o(1)}{\|\Delta_m(u,v,j)\,(p_2 \times p_3) + o(1)\|} \to \frac{p_2 \times p_3}{\|p_2 \times p_3\|} := n(p)
\end{aligned} \tag{5.54}$$

which shows the continuity of the normal to the surface at the point p. $\square$

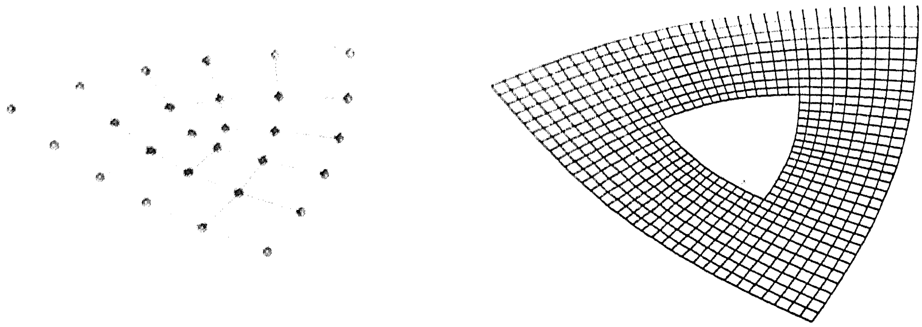


Figure 5.22: The vector V (diamonds) and the characteristic map Ψ_0 for $n = 3$

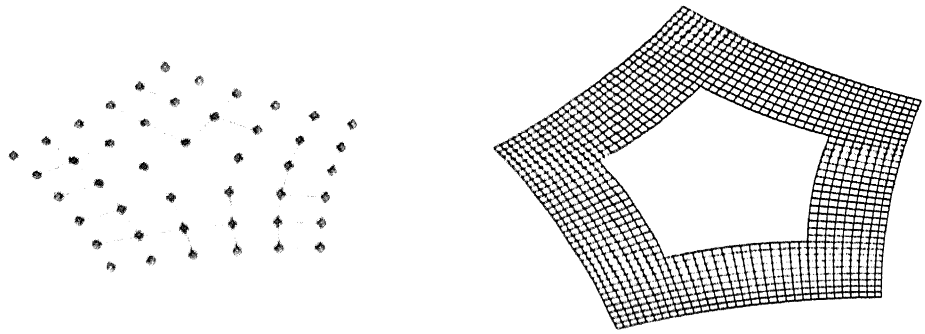


Figure 5.23: The vector V (diamonds) and the characteristic map Ψ_0 for $n = 5$

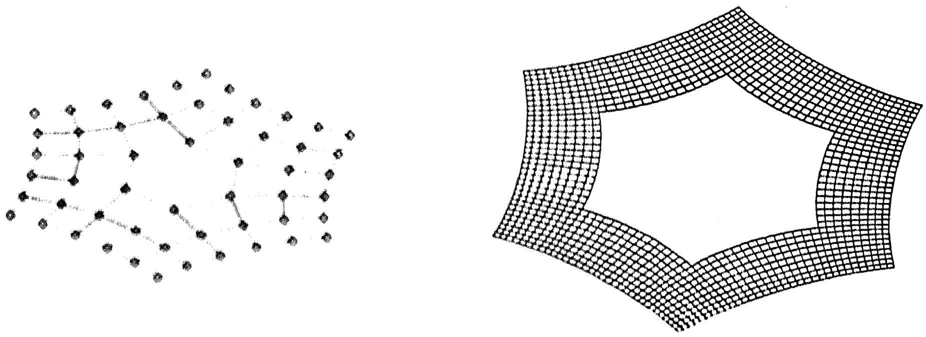


Figure 5.24: The vector V (diamonds) and the characteristic map Ψ_0 for $n = 6$

5.9 Applications and Examples

In this section we give some examples illustrating our subdivision scheme. One of the advantages of the scheme is that taking different values for h we limit surfaces of different sizes. Thus h is used as a design parameter.

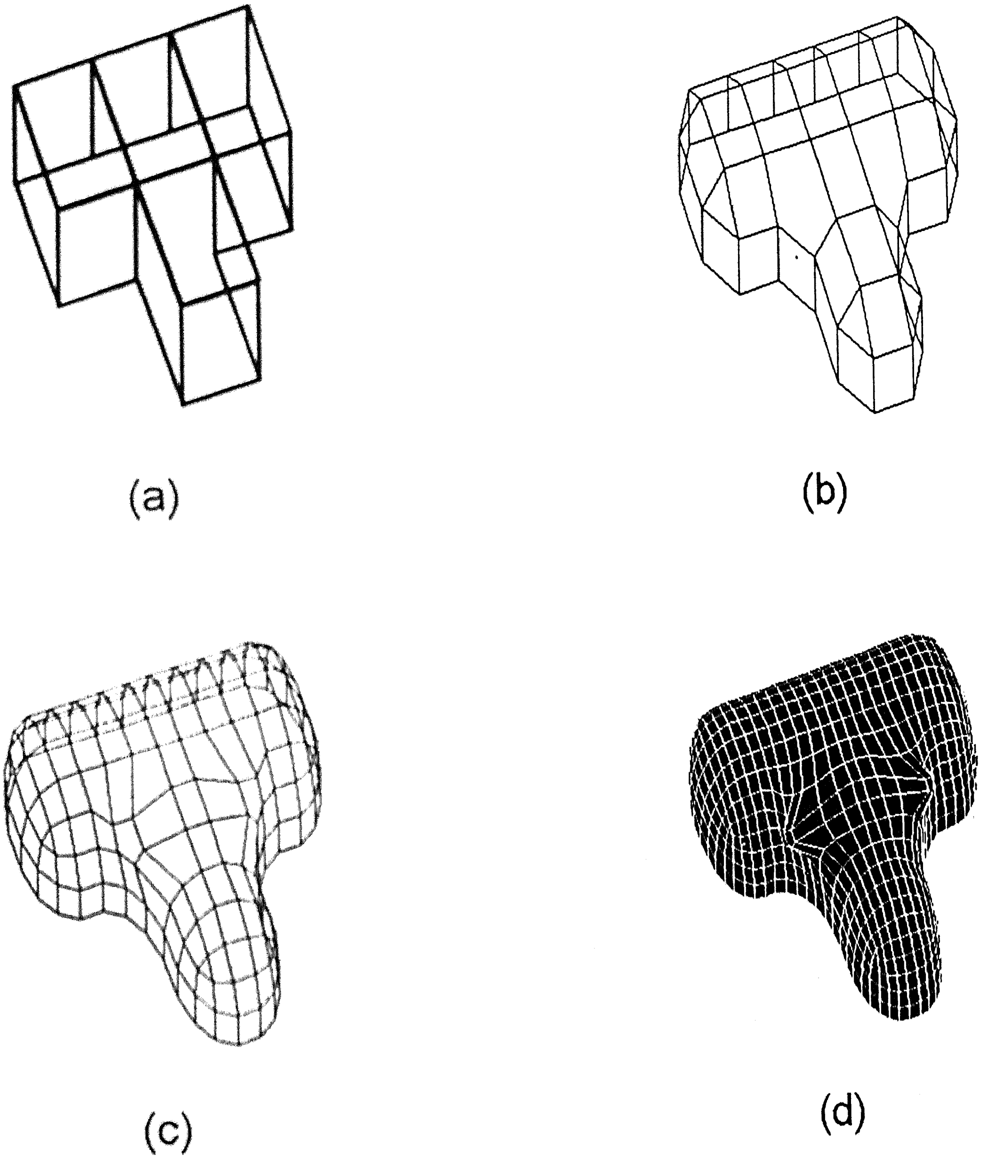


Figure 5.25: (a) Original topology. Limit surfaces after (b) First (c) Second (d) Third iterations

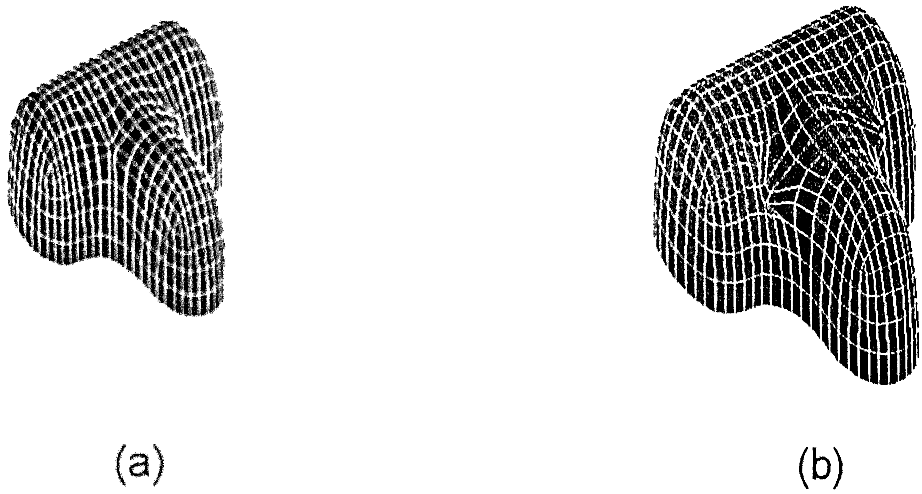


Figure 5.26: Comparison: (a) Doo (b) Nonstationary, $h = 0.5$.

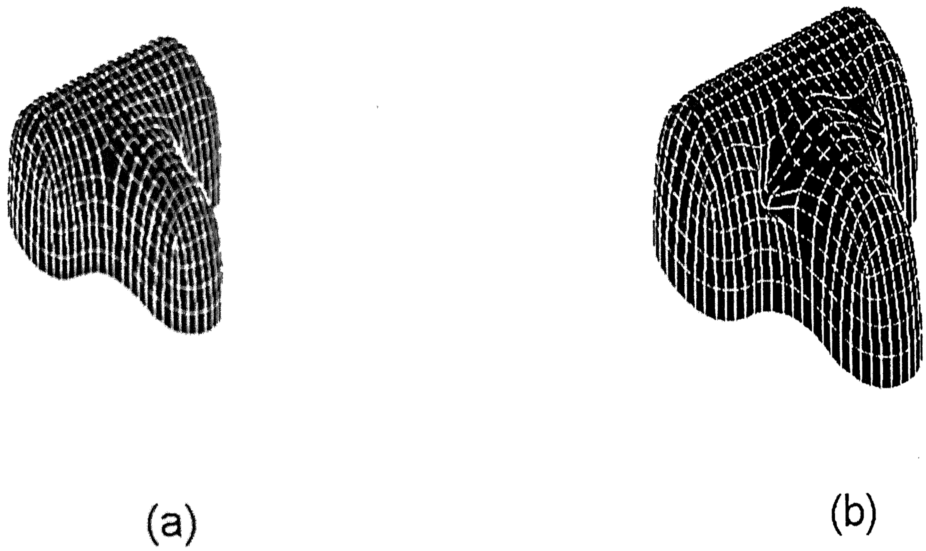
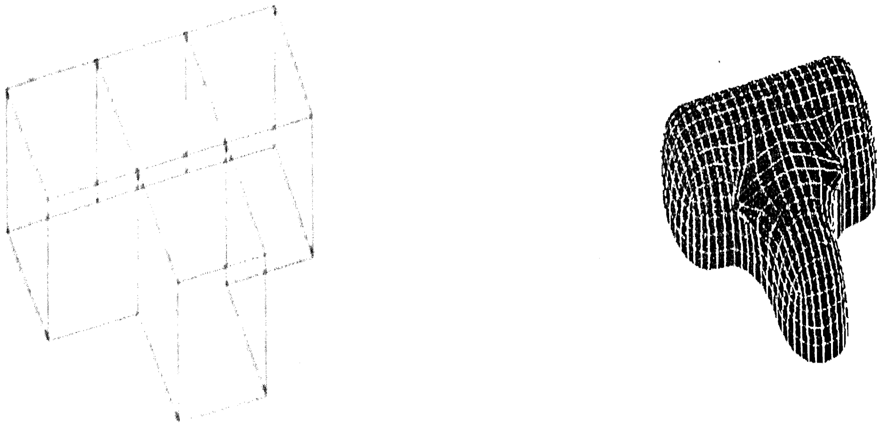
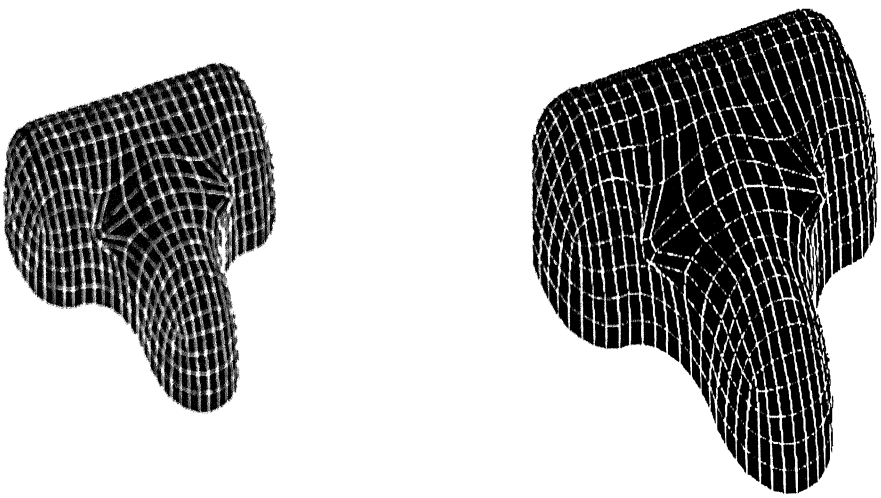


Figure 5.27: Comparison: (a) Catmull (b) Nonstationary, $h = 0.5$



(a) (b)

Figure 5.28: Different h: (a) Original (b) h=0.25



(a) (b)

Figure 5.29: Different h: (a) h=0.5 (b) h=0.75

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Errata

Page No.	The line +: from the top -: from the bottom	Mistake, imperfection	Correction, modification
v,	11^+	of coefficients	of coefficients which is a finitely supported real sequence
vi,	5^-	of polynomial spline	for polynomial splines
vii,	5^-	Dyn and Levin	Dyn, Levin and Gregory
1,	8^+	find	finding
2,	9^-	spline	splines
3,	2^-	so called	the so called
3,	6^-	stable	stable and
3,	11^-	inherits	inherit
5,	3^-	N. Dyn	N.Dyn et.al.
6,	4^+	Cavaretta and Micchelli	Cavaretta, Dahmen and Micchelli
7,	2^+	looking .... spline	looking at ..., spline
7,	8^+	algorithm of	algorithm for
7,	11^+	methodological ...	mathematical
7,	11^+	...Chapter	...chapter
7,	16^+	Dyn and Levin	Dyn,Levin and Gregory
7,	7^-	the thesis, scheme $\cdots$	the thesis, the
9,	7^-	BSS	Binary Subdivision Sch
10,	1^+ & 3^+	Theorem	theorem
10,	2^+	of the same support	of the same
15,	11	or trigonometric	or the trigonometric
17,	9^+	translations	translates
17,	5^-	solutions	elements
17,	4^-	$a, b \in \mathbb{R}$	$a, b \in \mathbb{R}$ be

Page No.	The line +: from the top -: from the bottom	Mistake, imperfection	Correction, modification
23,	10^{+}	$\cdots, x')$	$\cdots, x')$.
23,	6^{-}	$\cdots, x')$	$\cdots, x')$.
24,	9^{-}	be	to be
25,	4^{-}, 7^{-}	$[t_r, t_s]$	$[t_r, t_s]$,
25,	6^{-},	$[t_r, t_s]$	$[t_r, t_s]$.
40,	11^{-},	are	be
52,	9^{-},	mask	masks

Response to examiner's queries

Suggestion no 1: " + Theorem 1.4 : Block-circulant matrices are not necessarily diagonizable In particular, the blocks themselves are not necessarily circulant. Hence $F_m \hat{A}_j F_m^*$ does not make sense. The best one can hope for is a transformation to block diagonal form, but this is sufficient for all application in scope."

Response:

Theorem 1.4 (Diagonalization of a block circulant matrix) A block circulant matrix $A = bcirc(A_0, A_1, \ldots, A_{n-1})$ has the block diagonalization form

$$A = (F_n \times F_m)^* diag(F_m \hat{A}_0 F_m^*, \ldots, F_m \hat{A}_{n-1} F_m^*)(F_n \times F_m).$$

Suggestion no 2: "+ Page 25 : This is my serious observation. The weights for computing the new points a_i do not sum up to 1. Thus the rule is not affine invariant and therefore not meaningful in a goemetrical sense (results depend on the specified coordinate system). I am sure that the artifacts which are visible in Figures 5.25 through 5.29 are due to this problem. The choice of weights has to be corrected, and it should be explained how this choice is motivated (something quite vague"natural generalization of the weights for the regular case" will be sufficient)."

Response:

This is a non-stationary scheme. Thus the sum of the mask computing newpoints a_is are no equal to one. Instead sum of the masks at mth level of iteration is equal to $r_m = \frac{\cos^2(h/2^{m+1})}{\cos^2(h/2^m)}$ The choice of weights for the masks is a natural generalization of the weights for the regula case. Since $r_0 r_1 \cdots r_\infty = \frac{1}{\cos^2(h)}$, we normalize our scheme by multiplying $\cos^2(h)$ to th limit surfaces of the non-normalize scheme. As a result our scheme has now convex hu property and translation invariant property. Thus they are now useful in geometrical sens We are not normalizing the weights of the mask since some of the techniques of conve gence analysis. adopted in Section 5.7 may not follow for the normalized weights. We a describing the properties of limit surfaces of the normal scheme in detail in a separate sectic

1 Properties of limit surfaces

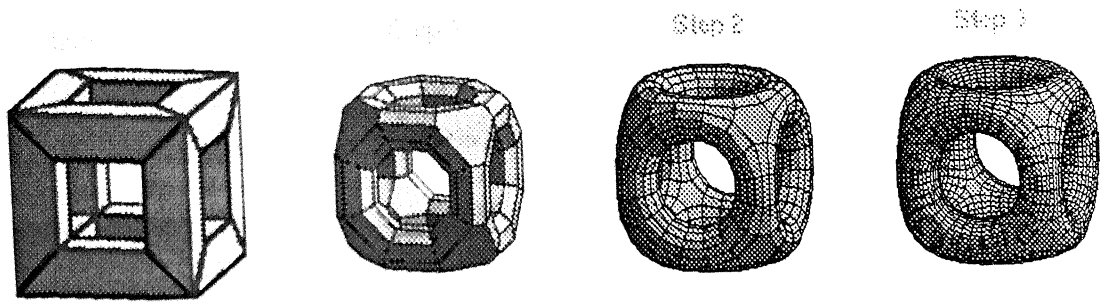


Figure 1: *Original control mesh and three iterations of our nonstationary subdivision scheme (normalized) for $h = 0.698$.*

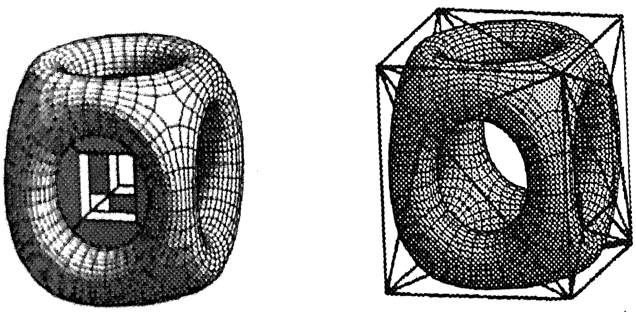


Figure 2: *Convex Hull property,* $h = 0.698$.

Our scheme is translation invariant. To show this consider the initial control mesh $\mathcal{F}^0$ and its regular part U_0 as discussed in Section 4. The initial set of control points B_0, written in the form of a block matrix, consists of vertices of $\mathcal{F}^0$ as well as U_0. We translate the initial control mesh by a vector $t \in \mathbb{R}^3$. As a result we get a new control mesh whose regular part U_0' satisfies $U_0' = U_0 + t$ and a new block matrix $B_0' := B_0 + et$, corresponding to the new set of control points.

Let X and X' be the limit surfaces of the subdivision scheme $\{S_k\}$ for the control points U^0 and $U^{0'}$ respectively. It is observed in Section 2 that $\cos^2(h)X' = \cos^2(h)\,X + t$. Let $\mathbf{x}_m \in pr^m(X)$ and $\mathbf{x}_m{}' \in pr^m(X')$ be points on the m-th prolongation of X and X' respectively corresponding to a parameter value (u, v, j). Then

$$\begin{aligned}\mathbf{x}_m{}' &= b_m(u,v,j)M^{(m)}B_0{}' = b_m(u,v,j)M^{(m)}B_0 + b_m(u,v,j)M^{(m)}et \\ &= \mathbf{x}_m + b_m(u,v,j)M^{(m)}et.\end{aligned}$$

Since e is an eigenvector for all matrices M_m associated with the eigenvalues $\frac{\cos^2(h/2^m)}{\cos^2(h/2^{m-1})}$ we get $M^{(m)}et = \frac{\cos^2(h/2^m)}{\cos^2(h)}et$. By (4.7) we have $b_m(u,v,j)e = \frac{1}{\cos^2(h/2^m)}$ for all $(u,v,j) \in \Omega_m$. Therefore

$$\mathbf{x}_m{}' = \mathbf{x}_m + \frac{1}{\cos^2(h)}t.$$

Since the limit surface $\bar{P}$ is continuous, the extraordinary points are also translated by $\frac{1}{\cos^2(h)}t$. As a whole, all points in the limit surface $\bar{P}$ are translated by vector $\frac{1}{\cos^2(h)}t$ and hence the normalized limit surface $\cos^2(h)\bar{P}$ is translated by the vector t.

We also note that the normalized limit surface $\cos^2(h)\,\bar{P}$ lies in the convex hull of B_0 (see Figure 2).

In case control meshes haveing nonpositive solid angle the normalized limit surfaces show saddle point behaviour near extraordinary points. The Doo-Sabin subdivision surfaces in this case look better than our normalized limit surfaces(see Figure 3). But if all the vertices of the control meshes have positive solid angle, our normalized limit surface behave nicely(see Figure 4).

Finally we observe that our scheme is a dual scheme.

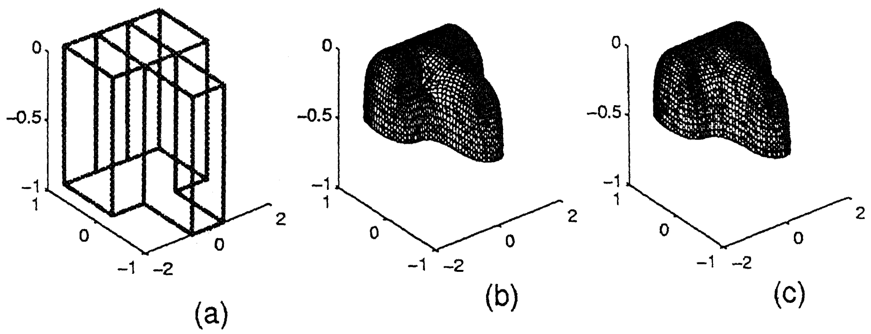
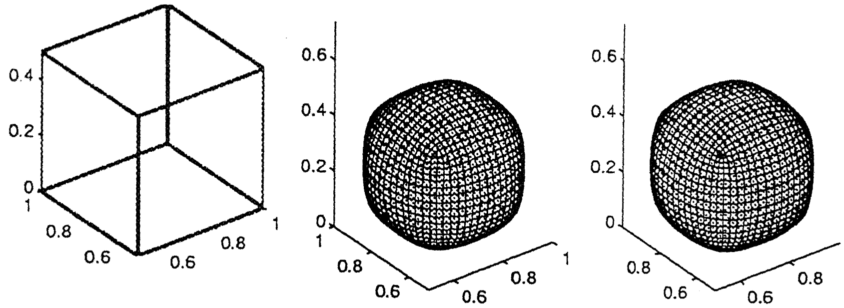


Figure 3: *Comparison. (b) Doo-Sabin (c) Our scheme,* $h = 0.5$.



(a) (b) (c)

Figure 4: *Comparison. (b) our scheme, $h = 0.5$ (c) Doo-Sabin.*

Suggestion no 3: "+ above theorem 5.7 : The statements of the linear independce of the vector $\{\partial/\partial u\ x_m\}$ and $\{\partial/\partial v\ x_m\}$ is not proven. I understand that such a proof would be very laborious (and perhaps not woth do be carried out in detail), but a figure showing the parameter lines of the characteristric map might be used to support the claim."

Response:

Linear independence of $\partial/\partial u\ x_m$ and $\partial/\partial v\ x_m$ is required for showing $\Delta_m(u,v,j) := det\frac{\partial\Phi_m(u,v,j)}{\partial(u,v)} \neq 0$. A figure showing parameter lines of the characteristic map might sup port this claim. But we are providing figures which shows part of the surfaces Δ_0 for th cases $n = 3, 5$ and $n = 6$ respectively. The surface Δ_0 consist of $3n$ patches. The figur below shows $3n$th, first and second patch of Δ_0 for the cases $n = 3, 5$ and $n = 6$ respectivel They are captioned as Jn. Clearly, these patches never intersect the plane $z = 0$. Becau of block symmetry of the matrices M_1 other patches of Δ_0 behave identically, i.e., they a nonzero.

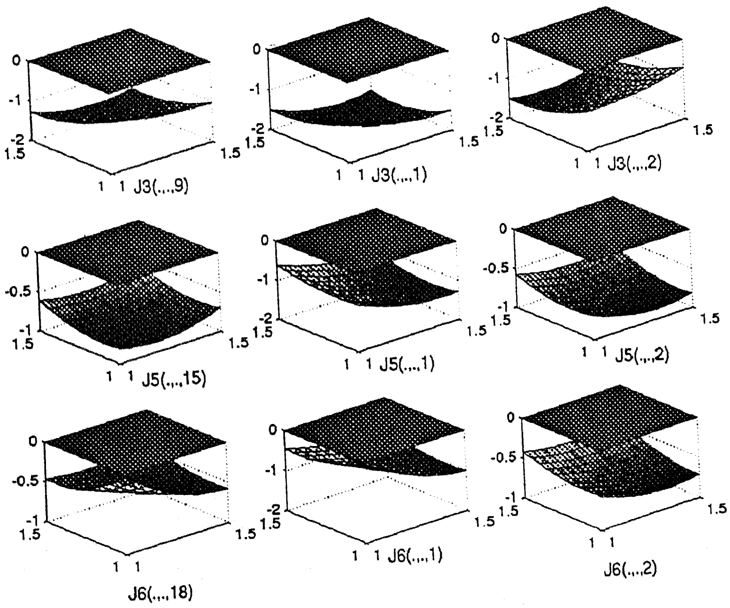


Figure 5: The Surface Δ_0